BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review
of Mathematical Science

EDITED BY

EARLE R. HEDRICK
DAVID R. CURTISS — WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

THEOPHIL H. HILDEBRANDT — TULLIO LEVI-CIVITA
HARRY W. KUHN — HENRY L. RIETZ
DERRICK N. LEHMER — CAROLINE E. SEELY
J. H. M. WEDDERBURN

VOLUME XXXIV, NUMBER 1

JANUARY-FEBRUARY, 1928

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1928

PUBLISHED BIMONTHLY

Whole No. 352 — $7.00 a Year

Entered as second class matter at the post office at Menasha, Wis., May 13, 1926.

JOURNALS OF THIS SOCIETY

For complete price list, see the July-August issue.

I. THE TRANSACTIONS
To members, $5.25; to others, $7.00.

II. THE BULLETIN
To members, free; to others, $7.00.

III. THE AMERICAN JOURNAL OF MATHEMATICS
Published by this Society and the Johns Hopkins University. To members, $5.25; to others, $6.00.

THE COLLOQUIUM PUBLICATIONS

ANNOUNCEMENT OF NEW VOLUMES

G. C. Evans, **The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems.** 1927. 150 pp. To members, $1.50; to others, $2.00.

E. T. Bell, **Algebraic Arithmetic.** Nearly ready. To members, $2.00; to others, $2.50.

L. P. Eisenhart, **Non-Riemannian Geometry.** In preparation.

G. D. Birkhoff, **Dynamical Systems.** In preparation.

These volumes are bound in cloth, and are each of about 200 pp. For earlier issues of this series, see a flyleaf of the May-June issue.

A. B. Coble, E. R. Hedrick, W. R. Longley *(Chairman)*
Committee on Sales of Publications

Orders may be addressed to
501 WEST 116TH STREET, NEW YORK, N.Y.

On sale also by the following official agents of this Society:

The Open Court Publishing Co., 337 E. Chicago Ave., Chicago, Ill.

Bowes & Bowes, 1 Trinity St., Cambridge, England.

Hirschwaldsche Buchhandlung, Unter den Linden, 68, Berlin.

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

EARLE R. HEDRICK
DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

THEOPHIL H. HILDEBRANDT TULLIO LEVI-CIVITA
HARRY W. KUHN HENRY L. RIETZ
DERRICK N. LEHMER CAROLINE E. SEELY
J. H. M. WEDDERBURN

VOLUME XXXIV

JANUARY TO DECEMBER, 1928

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1928

The Collegiate Press
GEORGE BANTA PUBLISHING COMPANY
MENASHA, WISCONSIN

BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

THE OCTOBER MEETING IN NEW YORK

The two hundred fifty-seventh regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 29, 1927, extending through the usual morning and afternoon sessions. Attendance included the following ninety-five members of the Society:

Alexander, Allison, R. C. Archibald, R. G. Archibald, J. W. Arnold, A. A. Bennett, Benton, Bratton, Brinkmann, R. W. Burgess, Carrié, E. T. Carroll, Coble, Cramlet, Edmondson, Eisenhart, Engstrom, Fenn, Fiske, Fite, D. A. Flanders, Forsyth, Fort, M. C. Foster, Frink, Garver, Gehman, Gill, Glenn, P. H. Graham, Gronwall, C. C. Grove, Guggenbühl, Hebbert, L. S. Hill, Hille, Hofmann, Hoyt, Huber, Dunham Jackson, Joffe, R. A. Johnson, Kasner, Kholodovsky, Kline, Koopman, Kormes, Lamson, Langman, Lefschetz, MacColl, S. P. Mead, Meder, H. H. Mitchell, Richard Morris, Mullins, Murnaghan, C. A. Nelson, Newman, K. E. O'Brien, Ore, Paradiso, Pfeiffer, Pierpont, Post, R. G. Putnam, Reddick, R. G. D. Richardson, D. E. Richmond, Ritt, Robertson, Rutledge, Schoonmaker, Seely, Siceloff, Simons, Slotnick, Smail, P. A. Smith, Virgil Snyder, Stetson, Tamarkin, J. E. Thompson, Veblen, H. E. Webb, Wedderburn, Weida, Weisner, A. P. Wheeler, Whittemore, Widder, W. A. Wilson, Margaret M. Young, Zippin, Zobel.

The Secretary announced the election of the following persons to ordinary membership in the Society:

Mr. Abraham Adrian Albert, University of Chicago;
Dr. Ralph George Archibald, Columbia University;
Mr. Armond W. Bear, Marquette University;
Mr. Richard Stevens Burington, Case School of Applied Science;
Professor Harold Hardesty Downing, University of Kentucky;
Mr. Walter Henry Gage, Victoria College, Victoria, British Columbia;
Mr. Ellis Richard Heineman, Michigan Agricultural College;
Mr. Isaac Newton Heminger, San Jose, Calif;
Professor Edward Milton Little, University of Montana;
Professor Ernest Lloyd Mackie, University of North Carolina;
Professor Marion Lee MacQueen, Southwestern College;
Professor Max Morris, Case School of Applied Science;
Mr. Kenneth Ernest Moyle, High School, Somerville, N. J.;
Professor Oystein Ore, Yale University;
Mr. Rafael Sanchez-Diaz, University of Porto Rico;

Miss Mildred Ellen Taylor, Knox College;
Mr. Carleton Russell Worth, University of Arkansas;
Professor Frank Lynwood Wren, George Peabody College for Teachers;

Nominees of the Georgia Power Company as sustaining member:

Mr. C. E. Bennett, Georgia Power Company, Atlanta;
Mr. S. C. Bleckley, Georgia Power Company, Atlanta;
Mr. F. A. Brine, Georgia Power Company, Atlanta;
Mr. W. A. Hammel, Georgia Power Company, Atlanta;
Mr. W. P. Hammond, Alabama Power Company, Birmingham.

At the meeting of the Board of Trustees, the editors of the Bulletin and of the Transactions were authorized to sign printing contracts with the Banta Publishing Company.

The council fixed the dates for regular New York meetings of the Society in New York during 1928 as follows: February 25, October 27, and, in place of the meeting usually held on the last Saturday in April or the first Saturday in May, a two-day meeting on April 6–7, the Friday and Saturday before Easter. An invitation for the summer meeting of 1931 was received from the University of Minnesota.

On nomination of the Western group, a committee consisting of Professors E. W. Chittenden (chairman), L. M. Graves, and M. H. Ingraham was appointed to choose a Symposium speaker for the Easter, 1928, meeting at Chicago.

Nominations for officers and other members of the Council was adopted, and ordered printed on the official ballot.

President Snyder presided at the morning session. In the afternoon two sectional meetings were held; Dr. Gronwall presided over the section of Analysis and Point Sets, at which papers numbered 11–26, 46–48 were read, and Professor Pierpont over that of Algebra and Geometry, at which papers numbered 27–45 were read.

At the request of the Program Committee, Professor F. D. Murnaghan delivered an address at the end of the morning session, entitled *Modern hydrodynamical theory, with special reference to aeronautics*. This address will appear in full in an early number of this Bulletin. Titles and abstracts of the other papers read at this meeting follow below. The papers

of Ayres, Bray, Copeland, Douglas, Frink (second and third papers), Garver (third paper), Hickey, Hollcroft, Michal, Roos, Seely, Stetson, Weatherburn, and Whyburn were read by title. Mr. Gourin and Mr. Wang were introduced by Professor Ritt, Miss Hickey by Professor Evans, and Professor Weatherburn by Professor Kasner.

1. Professor C. H. Forsyth: *The amount of a single sum of money at any number of rates of interest.*

In the issue of this Bulletin for June-July, 1921, the author presented certain generalizations of some of the most fundamental formulas of the mathematical theory of finance. Those generalizations were based upon the use of two and three rates of interest. The purpose of the present paper is to show the nature and form of the function which is obtained when the number of rates of interest is increased indefinitely.

2. Professor C. H. Forsyth: *The determination of monthly items from yearly items of the secular trend of a time series when that trend is assumed to be other than linear.*

In the case of a linear trend, a straight line is fitted to yearly items or totals of a time series, and the coefficients of the linear equation divided by 12 give the monthly items. Occasionally, the trend is clearly other than linear; then the yearly items are easily obtained when once the type of curve is selected, but the corresponding monthly items cannot be found by simple division. The author shows they may be found by interpolation.

3. Dr. H. P. Robertson (National Research Fellow): *An alternative to De Sitter's cosmology.*

The 4-dimensional manifold defined by the line element $ds^2 = -e^{2kct} \cdot (dx^2+dy^2+dz^2)+c^2dt^2$, $k>0$, is of constant Riemannian curvature, and is therefore mathematically equivalent to De Sitter's universe. Interpreting the variable t occurring in this form as the proper-time of an observer at a point x, y, z=constant, the "mass-horizon" and "arrest of time at the horizon" paradoxes of De Sitter are obviated. Because of the dynamical form of the line element, however, natural processes are not reversible; in particular, a shift of spectral lines in light from distant objects is predicted which is in accordance with existing data on extra-galactic nebulae and which leads to a value $k=5 \cdot 10^{-28}(cm)^{-1}$. The universe may be represented as a hypersphere in a flat 5-space, hence the relations existing between freely moving observers (the analogue of the Lorentz transformations of special relativity) are equivalent to the rotations of flat 5-space.

4. Dr. H. P. Robertson: *Conditions under which a certain type of solution of Schrödinger's wave equation can be found.*

The conditions under which the fundamental equation of Schrödinger's "wave-mechanics" is solvable by separation of variables are found for the

case of a conservative system whose kinetic energy is expressed in orthogonal coordinates. It is shown that in addition to the conditions given by P. Stäckel for the solution of the corresponding Hamilton-Jacobi equation a certain functional relation must be satisfied.

5. Dr. L. S. Hill: *Certain matrices in finite algebraic fields, and their relation to code telegraphy.*

Let a matrix in which no determinant of any order vanishes be called, for convenience, uniform; the present paper directs attention to the bearing of such matrices, constructed in various finite algebraic fields, upon certain important problems of accuracy assurance in the transmittal of electrical communications,—telegraph, cable, radio. A special type of uniform matrix, which appears to be directly amenable to practical application, is treated with particular care. It would be easy to construct a mechanical device, similar to the ordinary adding machine but somewhat less complex, which would automatically accomplish all checking operations based upon this special type. The paper leads to a further problem which is of purely algebraic interest, and which will be studied later.

6. Professor D. E. Richmond: *A complete set of relations between type numbers of extremals joining a pair of points.*

Let A and B be a pair of points in an extremal-convex region for a calculus of variations problem. An extremal arc AB may be assigned a type number n equal to the number of points on the arc conjugate to A. Let M_n represent the number of extremals of type n joining A to B. The author has proved the completeness of a certain set of number relations between the integers M_n, when no type number n exceeds 2. When extremals of type 2 exist joining A to B, the relations are $M_0=2$, $M_0-M_1+M_2=1$. It has been shown that there exists a calculus of variations problem and a pair of points A and B such that the number of extremals of type i ($i=1, 2, 3$) is M_i, where the M_i's are arbitrary positive integers satisfying the relations stated.

7. Dr. A. H. Copeland: *Admissible numbers in the theory of probability.*

The problem of establishing the consistency of the fundamental assumptions of the theory of probability is proved in this paper to be equivalent to that of showing the existence of certain sets of numbers, $A(p)$. These assumptions are consistent in the case of a given probability, p, if and only if the corresponding set, $A(p)$, is non-empty. Admissible numbers are the numbers that belong to the sets $A(p)$. Every admissible number must satisfy an infinite set of independent equations and hence the consistency of the fundamental assumptions seems doubtful. Not only is the consistency of the assumptions open to question, but also the customary definition of independence is so loosely stated that, if one takes the most obvious interpretation of this definition, the theorem of compound probability is false. However, it is shown that this definition of independence

can be so stated that this theorem is true. It is also proved that the sets $A(p)$ are non-empty and hence that the fundamental assumptions are consistent. Moreover, it is proved that each point x of the interval $0<x<1$ is a point of condensation of each of the sets $A(p)$ where $0<p<1$.

8. Dr. A. H. Copeland: *Stability of the motion of a gyroscope in contact with a smooth horizontal plane.*

The problem considered in this paper is the stability of the motion of a gyroscope which is a smooth body of revolution and which is acted upon only by gravity and a constraint which keeps it in contact with a smooth horizontal table. The motion is described in terms of a curve Γ traced by the axis of the gyroscope on the unit sphere whose center is at the center of mass of the gyroscope and whose orientation is fixed in space. The north pole is taken at the top of the sphere. The curve Γ is bounded by two parallels of latitude. It is proved that for slight changes in the initial conditions large changes can occur in the positions of these parallels, that is, the motion can be unstable; but these unstable motions can be cut out by giving the gyroscope a sufficiently large spin. The method used is an extension of the method employed in the author's paper entitled *Types of motion of the gyroscope* (see abstract No. 10, below).

9. Dr. A. H. Copeland: *The motion of a gyroscope with variable spin.*

This paper is a discussion of the motion of a gyroscope which is acted upon by gravity, by a constraint which keeps one point on its axis fixed in space, by a torque, S, which tends to increase or decrease the spin of the gyroscope about its axis, and by no other forces. The case in which S is identically zero gives a classical problem. In this paper it is assumed that S is a function of the time, the position of the gyroscope, and the velocities of the coordinates determining the position. The effect of S is traced throughout the entire motion. The major portion of this paper is devoted to the case in which the spin is monotone and nowhere stationary. All the possible types of motions of gyroscopes with always increasing spins, or always decreasing spins, and all the possible changes in these types are determined. It is also shown that some types of motion are unstable and break down into other types when small changes are made in the spin.

10. Dr. A. H. Copeland: *Types of motion of the gyroscope.*

A method of graphical representation of gyroscopic motion is developed in this paper. A simple representation is given of the complete history of all motions of all gyroscopes subject to the following restrictions: each gyroscope is acted upon by gravity and by a constraint which keeps one point on its axis fixed in space, and by no other forces. With the aid of Osgood's intrinsic equations, this history is extended to include new intrinsic properties of the space cone traced by the axis of the gyroscope. By means of these methods it is possible to list the properties of the motion

more fully than has hitherto been done, exhibit their dependence upon the initial conditions, and show how one type of motion changes into another as the initial conditions are varied continuously.

11. Professor Edward Kasner: *Non-monogenic (or polygenic) functions.*

The author proposes the term *polygenic* for arbitrary functions $w=\phi(x, y)+i\psi(x, y)$ of the complex variable $z=x+iy$ where the derivative depends on the slope m as well as the point z. To each point z we have then ∞^1 values of $dw/dz=\alpha+i\beta$, which when plotted as points in a new plane (α, β) form a circle called the *derivative circle.* Thus to the entire polygenic function corresponds a *congruence* of circles. This in special cases degenerates; e.g., if the given function is monogenic the circles all become points; for a function of $x-iy$, we obtain in the (α, β) plane ∞^1 circles with centers at the origin; if the centers all lie on the axis of α, the function must obey merely one of the Cauchy-Riemann equations, etc. The *mean derivative* of any polygenic function is defined as $H+iK$, where H, K are the coordinates of the center of the derivative circle. The mean derivative is monogenic when and only when ϕ and ψ obey the Laplace equation. The transformation is then termed general harmonic by the author; this includes conformal as a very special case. The point (α, β) on any derivative circle always moves, as m changes, with angular velocity twice that of m and in the opposite direction. For each circle a certain initial radius must be fixed; the complete graph is thus a congruence not of circles but of *clocks*, each clock with four coordinates. A general theory of such congruences is given.

12. Professor J. F. Ritt: *Certain points in the theory of Dirichlet series.*

The results of this paper, all of which deal with Dirichlet series, bear upon the following topics: Picard's theorem, integral functions, asymptotic convergence. The first part contains generalizations of the theorems of Landau and Schottky. The second part deals with integral functions defined by everywhere absolutely convergent Dirichlet series, and develops the relations between the growth of the modulus of the function and the coefficients of the series. The third part carries over to Dirichlet series the theory of asymptotic convergence of power series.

13. Dr. B. O. Koopman: *Some theorems on inversion.*

Consider the substitution (1). $y_i=f_i(x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$, where the f's are analytic functions of $(x_1, \cdots, x_n)=(x)$ for $(x)=(0)$. This substitution has only been studied completely when the Jacobian J does not vanish for $(x)=(0)$, when the transformation has a unique analytic inverse. The object of our paper is to investigate the substitution in the case where $(x)=(0)$ is a non-specialized point of the complex locus $J=0$. By means of non-singular substitutions of the variables (x) and (y) we reduce the substitution (1) to a standard form, from which it appears

that it may be inverted by means of algebroid functions. The results are then applied to the extension of the theorem of functional dependence. (See Osgood, *Lehrbuch der Funktionentheorie*, II, chap. 2, §§19, 20, 23.)

14. Mr. Eli Gourin: *On irreducible polynomials.*

Given an irreducible polynomial $Q(y_1, y_2, \cdots, y_p)$ of p variables $y_1, y_2, \cdots, y_p$, the object is to determine all positive integers $t_1, t_2, \cdots, t_p$ for which the polynomial $Q(y_1^{t_1}, y_2^{t_2}, \cdots, y_p^{t_p})$ is reducible.

15. Mr. Eli Gourin: *Periodic Weierstrass zeta functions.*

This paper presents a study of conditions for which Weierstrass's zeta function becomes a singly periodic function.

16. Mr. S. C. Wang: *On a linear addition theorem in three arguments.*

It is shown in this paper that all continuous solutions of the functional equation of real variables $f(x+y+z)=f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z)$ are of the form $f(t)=At^2+Bt$, where A and B are two independent arbitrary constants.

17. Dr. T. C. Benton: *On continuous curves homogeneous except for certain points.*

This paper proves the following theorems: (1) A point set M is homogeneous except for n points $c_1, c_2, \cdots, c_n$ if for any pair of points a_1, a_2 distinct from all c_i a continuous (1-1) correspondence ψ exists such that $\psi(M)=M$ and $\psi(a_1)=a_2$; and for every i there exists a point x_i such that no correspondence π exists such that $\pi(M)=M$ and $\pi(x_i)=c_i$. (2) A bounded continuous curve homogeneous except for one point c_1 is a countable set ($\geqq 2$) of simple closed curves having only c_1 in common and such that only a finite number of them are of diameter greater than any given positive number. (3) A bounded continuous curve homogeneous except for c_1 and c_2 is a finite number ($\neq 2$) of arcs joining c_1 to c_2. (4) A bounded continuous curve homogeneous except for c_1, c_2, c_3 is either two sets homogeneous except for two points which have only one c_i in common and have the same number of arcs, or three such sets joined cyclically with the same number of arcs in each set. (5) An unbounded continuous curve homogeneous except for c_1 is a finite number of rays from c_1. (6) An unbounded continuous curve homogeneous except for c_1, c_2 does not exist.

18. Professor H. M. Gehman: *On extending a continuous* (1-1) *correspondence.* Second paper.

In continuation of results recently published (Transactions of this Society, vol. 28 (1926), pp. 252–265), it is proved that if any two plane point sets which are closed and bounded and which have the properties that (1) every maximal connected subset is a continuous curve, and (2) not more than a finite number of maximal connected subsets are of diameter greater than any positive number, are in continuous (1-1) correspondence in such a way that the correspondence preserves sides *in the*

same sense, then a continuous (1-1) correspondence of their planes can be defined which is identical with the given correspondence for points of the two sets.

19. Professor W. A. Wilson: *On bounded regular frontiers in the plane.*

This paper appears in full in the present number of this Bulletin.

20. Professor R. G. Putnam: *Note on countable sets of points of non-separable metric spaces.*

It is shown in this note that every countable set of points of a non-separable metric space is homeomorphic with a linear set.

21. Dr. W. L. Ayres: *An elementary property of bounded domains.*

This paper will appear in full in an early number of this Bulletin.

22. Dr. W. L. Ayres: *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve.*

Let H denote the sum of the boundaries of the complementary domains of a continuous curve M in a plane S. These theorems are proved: (1) if an M-domain is simply connected with respect to M, there exists a simply connected domain R such that $R \cdot M = D$; (2) if every two points x and y of H may be connected through $x+y$ and $S-M$, then H is a continuous curve and either (a) H is the boundary of one of the complementary domains of M, or (b) M consists of three arcs with common end points and no two having any other point in common, or (c) M consists of three rays with the same vertex and no two having any other only its common, or (d) M consists of an open curve plus an arc having point in end points in common with the open curve; (3) if every two points x and y of M may be connected through $x+y$ and $S-M$, then $M=H$; (4) if every three points of H lie together on some complementary domain of M, then H is the boundary of one of the complementary domains of M.

23. Dr. G. T. Whyburn and Dr. W. L. Ayres (National Research Fellow): *On continuous curves in n dimensions.*

This paper will appear in full in an early number of this Bulletin.

24. Professor A. D. Michal: *Invariant functionals of functional forms.*

In the theory of algebraic invariants of algebraic forms in n variables, the underlying group is the general group of linear homogeneous transformations with non-vanishing determinants. This paper develops the first essentials of a corresponding theory in the space of real continuous functions of a real variable x. The "algebraic" *functional* forms are restricted to normal types and the fundamental group is taken to be the

group of all linear functional Fredholm transformations for which unity is not a characteristic value. "Tensor" invariant functionals of the coefficients of the functional forms are considered as well as scalar ones. The laws of transformation of the coefficients of the functional forms incidentally give rise to a theory of certain interesting integral equations which are not of the classic types.

25. Professor A. D. Michal: *Affinely connected function space manifolds.*

The first part of this paper develops a theory of functional tensors in the space of real continuous functions of a real variable x. The analogues of affine connection, infinitesimal parallelism, covariant differentiation, geodesics, normal coordinates, and tensor extension of n-dimensional geometry are then considered. The functional invariants of the functional affine connection arise as the integrability conditions of functional equations with funcional derivatives. The "geodesics" satisfy non-linear integro-differential equations of static type. The theory of the Fredholm integral equation underlies the theory of functional normal coordinates. The paper closes with a discussion of functional quadratic differential forms followed by a development of a theory of "Riemannian" function space manifolds. This paper combined with a part of the preceding paper will appear in the American Journal of Mathematics.

26. Miss Deborah M. Hickey: *The equilibrium point of Green's function for an annular region.*

This paper derives the necessary and sufficient condition for an equilibrium point in an annular region in terms of Fourier series and in terms of elliptic functions. Experimental investigations by W. S. Vaughn seemed to indicate that the equilibrium point remained fixed when the pole in the region was moved. A necessary and sufficient condition that this point be fixed is $2\zeta(\omega')-\omega'\wp(\omega')=0$. By means of Fourier series it is shown that this equation is satisfied for no ratio of the half periods ω, ω'. Nevertheless the experimental results are justified by obtaining theoretically the upper bound of the shift, which is very small.

27. Professor Oystein Ore: *Newton polygons in the theory of algebraic fields.*

The determination of the development of an algebraic function in power series corresponds to a certain extent to the determination of the prime-ideal decomposition of a prime p in an algebraic field. By generalizing the method of Newton polygons and by using the author's investigations on algebraic numbers (Mathematische Annalen, vols. 96 and 97) the complete decomposition of all primes p is obtained for every field given by an arbitrary equation.

28. Professor Oystein Ore: *Some relations between groups and ideals in Galois fields.*

As a generalization of the "Zerlegungsgruppe" of Hilbert, a group $G\{\mathfrak{A}\}$ is introduced corresponding to an arbitrary ideal $\mathfrak{A}$. The group $G\{\mathfrak{A}\}$ embraces all the substitutions for which the system of numbers in $\mathfrak{A}$ as ideal is invariant. The construction of $G\{\mathfrak{A}\}$ is determined, and some properties of the corresponding subfield studied.

29. Professor O. E. Glenn: *The invariancy of infinite series.*

The terms of a convergent infinite series $a, b, c, \cdots, \infty$ may be considered to be the coefficients of a polynomial f of order n in an infinite number of variables. Let f, with $n=1$ for example, be transformed by the infinite cyclic transformation $x=\lambda x'+\mu y'+\nu z'+\cdots\infty$, $y=\lambda y'+\mu z'+\nu t'+\cdots\infty, z=\lambda z'+\mu t'+\nu u'+\cdots\infty, \cdots$, where the series $\lambda, \mu, \nu, \cdots, \infty$ is convergent. Then certain series in the form of infinite polynomials in the terms $a, b, c, \cdots$ are relative invariants. The sum of the series is an invariant, the invariantive relation being $a'+b'+c'+\cdots=(\lambda+\mu+\nu+\cdots)\cdot(a+b+c+\cdots)$. The property of convergence is an invariant property. A theorem on the product of two series is a theorem on the transformation of one series by the infinite cyclic transformation determined by the other.

30. Dr. Louis Weisner: *Quadratic fields in which cyclotomic polynomials are reducible.*

For any particular value of m, the quadratic fields in which the polynomial whose zeros are the primitive mth roots of unity is reducible can be determined by methods described in the second volume of Weber's *Algebra*; but it does not seem practicable to carry out the computations for a general m. The present writer avoids these computations by proving two lemmas from which it is inferred that the cyclotomic polynomial is reducible in certain quadratic fields. The number of these fields is found to be precisely the number of intransitive subgroups of index 2 of the group of the cyclotomic equation relative to $R(1)$. The quadratic fields determined are the only ones in which the cyclotomic polynomial is reducible.

31. Dr. R. G. Archibald: *Diophantine equations in division algebras.*

This paper obtains necessary and sufficient conditions for the solvability in integers of diophantine equations, of the type $U^2-aR^2=(A^2-abB^2)\cdot(F^2-bC^2)$, which arise in an attempt to satisfy the associativity conditions for an algebra Γ based on a quartic equation $x^4+px^2+n^2=0$, irreducible in the field of rational numbers. The irreducibility of this equation imposes the condition that each of $a=2n-p$, $b=-2n-p$, ab be different from a perfect square. The associativity conditions for such algebras Γ were obtained by Dickson (*New division algebras*, Transactions of this Society, April, 1926). In the present paper numerical examples are given which satisfy all the conditions obtained.

32. Dr. Orrin Frink (National Research Fellow): *An algebraic method of differentiation.*

The theory of analytic functions of a hypercomplex variable in the case of certain familiar commutative linear algebras containing nilpotent

units is shown to lead to a method of obtaining the standard formulas of the differential calculus by purely algebraic means, without the use of limiting processes. The method can be used to find higher derivatives directly. It is shown that differentials may be treated as absolute infinitesimals when no higher derivatives are involved. In some cases the method can be used to find functions with prescribed addition theorems.

33. Dr. Orrin Frink: *A general type of algebras.*

The algebras studied here, which may be considered generalizations of Boolean algebras, are commutative and associative and obey the special laws $a+a=a$ and $a^2=a$ for all a's. Many examples are given, which are discussed with relation to supplementary postulates which some of them satisfy, such as $a+ab=a$, the dual of the distributive law, and the existence of identities of addition and multiplication. An important case is one in which the addition and multiplication operations are interpreted to mean least common multiple and greatest common divisor.

34. Dr. Orrin Frink: *Concerning the fundamental regions of elliptic functions.*

It is first shown that the only convex fundamental regions of elliptic functions are parallelograms and hexagons. Then the problem is solved of finding all possible sets of points in the plane such that a period parallelogram of fixed size, shape, and orientation will always contain exactly n of them no matter what its position in the plane.

35. Dr. Hazel E. Schoonmaker: *Non-monoidal involutions having a congruence of invariant conics.*

The purpose of this paper is to derive all birational involutorial point transformations of space which have the following properties: (a) each transforms every conic of a linear congruence into itself; (b) the transformations cannot be reduced birationally to the monoidal type.

36. Dr. Raymond Garver: *The binomial quartic as a normal form.*

This paper has appeared in the November-December number of this Bulletin.

37. Dr. Raymond Garver: *A rational normal form for certain quartics.*

This paper appears in full in the present number of this Bulletin.

38. Dr. Raymond Garver: *Tschirnhausen transformations on certain rational cubics.*

Normal forms for particular cubics are obtained, as well as a necessary and sufficient condition that a rational cubic may be transformed rationally into a binomial cubic.

39. Dr. Jesse Douglas (National Research Fellow): *The geometry of systems of K-spreads.*

This paper constructs for systems of K-spreads in N-space a theory analogous to the general geometry of paths ($K=1$) developed in a previous paper presented to the Society, May 7, 1927. If G is any group of coordinate transformations in the N-space, and H any group of parameter transformations on the K-spreads, we can have a G, H, geometry of K-spreads. This paper considers the affine, equivoluminar, and descriptive geometries of K-spreads, i.e., where G is the total group $\bar{x}^i=\bar{x}^i(x)$, and H is respectively (1) $u^\alpha=A^\alpha_\beta v^\beta$, (2) $u^\alpha=\phi^\alpha(v)$, where $\left|\partial u/\partial v\right|=$constant, (3) $u^\alpha=\phi^\alpha(v)$, ϕ^α arbitrary. For the affine geometry of K-spreads the fundamental partial differential equations are (4) $\partial^2x^i/\partial u^\alpha\partial u^\beta=H^i_{\alpha\beta}\,(x,\ \partial x/\partial u)$, where $H^i_{\alpha\beta}$ must satisfy the following condition: if $q^j_\alpha=A^\beta_\alpha p^j_\beta$, then $H^i_{\alpha\beta}(x,q)=A^\rho_\alpha A^\sigma_\beta H^i_{\rho\sigma}(x,p)$. A double tensor analysis arises in which Greek indices run from 1 to K, Latin indices from 1 to N. The formula for the affine connection is $\Gamma^i_{jk}=(K(K+1))^{-1}\partial^2H^i_{\alpha\beta}/\partial p^j_\alpha\partial p^k_\beta$. The integrability conditions for (4) are $B^i_{jkl}p^j_\alpha p^k_\beta p^l_\gamma=0$, where B^i_{jkl} is the curvature tensor based on the Γ's. A parameter transformation on the K-spreads converts (4) into $\partial^2x^i/\partial v^\alpha\partial v^\beta=H^i_{\alpha\beta}(x,\ \partial x/\partial v)+(\partial x^i/\partial v^\lambda)G^\lambda_{\alpha\beta}(x,\ \partial x/\partial v)$. The equivoluminar geometry of K-spreads is characterized by the vanishing of the contracted G's: $G^\lambda_{\lambda\beta}=0$.

40. Dr. Jesse Douglas: *One-to-∞^1 surface transformations of space.*

The equations (1) $X=X(x,\ y,\ z,\ p,\ q,\ a)$, $Y=Y(x,\ y,\ z,\ p,\ q,\ a),\cdots,$ $Q=Q(x,\ y,\ z,\ p,\ q,\ a)$ associate with each surface element $x,\ y,\ z,\ p,\ q$ a series of ∞^1 surface elements $X,\ Y,\ Z,\ P,\ Q$. The ∞^2 elements of any surface Σ (or union) are thereby converted, in general, into ∞^2 elements so distributed that there is one and only one through each point of a certain region of the $X,\ Y,\ Z$ space, thus defining a Pfaff equation in the $X,\ Y,\ Z$ space. Under what circumstances will the Pfaff equation corresponding to an arbitrary Σ be exact, so that the transformation (1) may be regarded as converting any surface Σ into ∞^1 surfaces? The present paper proves that this occurs when and only when there exists a transformation of the parameter a, (2) $a=a(x,\ y,\ z,\ p,\ q,\ b)$, such that the equations which result by substituting (2) in (1): (1') $X=\xi(x,\ y,\ z,\ p,\ q,\ b)$, $Y=\eta(x,\ y,\ z,\ p,\ q,\ b),\cdots,$ $Q=\kappa(x,\ y,\ z,\ p,\ q,\ b)$, define a *contact* transformation for each fixed value of b. As far as the present author knows, this result is not contained in Goursat's monograph *Le Problème de Bäcklund*, to the order of ideas of which the present paper belongs.

41. Dr. M. M. Slotnick (National Research Fellow): *A contribution to the theory of fundamental transformations of surfaces.*

Graustein has introduced a projective invariant of transformations F which we call the invariant C. The dual of this invariant, that is, the cross ratio in which a pair of corresponding tangent planes is divided by the

focal planes of the line of the harmonic congruence, is called the invariant H. These two invariants are in the relation $CH=e_1g/(g_1e)$, where e, g, e_1, g_1 are the coefficients of the second fundamental quadratic forms of the surfaces. A study of transformations F based on these two invariants yields dual fundamental theorems, and renders many facts easier to prove than by the classical methods. The perspective transformation for which $H=1$ enters as the dual of the radial transformation for which $C=1$; and in the same way the transformation $K(C=-1)$ and the transformation $\Omega(H=-1)$ are duals of one another. The theorems obtained are applied to transformations R in general, and also to those transformations R which are also K or Ω or both. Some attention has also been paid to transformations F of various special types when the nets are isothermal-conjugate.

42. Dr. M. M. Slotnick: *A method of applying tensor analysis to the study of rectilinear congruences.*

Some interesting results are obtained when the methods of tensor analysis are applied to deal with congruences of straight lines in a euclidean 3-space, the lines being represented by Study's "dual coordinates." The fundamental tensor is that of the linear element of the spherical representation of the congruence. We write $m_i=p_ia+q_i^la_l$, where m represents the coordinates of the middle point of the line of the congruence, a its direction cosines; and the subscripts on m and a represent differentiation with respect to the corresponding parameters. The coefficients of linear combination p_i and q_i^l are tensors which form the basis of the study. These are evaluated in terms of the fundamental tensor, and a second tensor whose components are the coefficients of the second fundamental quadratic form of the congruence.

43. Professor T. R. Hollcroft: *On nets of manifolds in i dimensions.*

The characteristics of a net in two dimensions have been fully determined, but little has been done in three dimensions and the author has been able to find nothing for dimensions greater than three. In this paper, the characteristics of a net of manifolds in any number of dimensions are obtained. This is done by establishing a (1, 1) correspondence between the lines of a plane and the manifolds of a net, and obtaining the characteristics of the branch-point curve. The number of manifolds of the net that have two hypernodes, one hyperbinode, two simple contacts or a stationary contact with another member of the net, are obtained. A multiple basis point of a net reduces all the characteristic numbers except the order of the jacobian. When the basis point is simple, this reduction occurs only for a net of curves in two dimensions. The amount of this reduction is found for any number of basis points of any given multiplicity.

44. Dr. J. M. Stetson: *Periodic conjugate nets.*

A net which can be transformed into itself by a sequence of Levy transformations, p of them using derivatives with respect to u, and q of

them using derivatives with respect to v, we call periodic (p, q). Such nets exist in spaces of $p+q-1$ or fewer dimensions. We find many types of transformations of nets periodic (p, q), all having theorems of permutability. Reciprocally derived nets and nets periodic under Laplace transformations seem to be the only cases hitherto considered. All the transformations known for these are simple special cases of the results of this paper.

45. Professor C. E. Weatherburn: *On curvilinear congruences.*

In an earlier paper (*On congruences of curves*, Tôhoku Mathematical Journal, 1927) the author has shown how the theory of curvilinear congruences in ordinary space of three dimensions may be extended along the lines followed for rectilinear congruences, making use of oblique curvilinear coordinates. The present paper contains a further extension of the theory.

46. Dr. C. F. Roos (National Research Fellow): *A dynamical theory of economics.*

In his recent papers in mathematical economics, the author has been primarily concerned with the mathematical aspects of the problems of the new dynamical economics, developed by G. C. Evans and himself, and references to existing economics papers have been avoided. It seems advisable, therefore, to give a history of these problems from the view point of the mathematical economist. In an attempt to do this, the present paper gives ample references to all existing related works known to the author, and shows how the work of Evans and the author is a generalization of existing theories of static equilibrium.

47. Professor H. E. Bray: *On uniform absolute continuity.*

Given a summable function $f(x)$ and a family of continuous functions $f(\mu, x)$ such that $\lim_{\mu=0} f(\mu, x)=f(x)$ almost everywhere; if $f(\mu, x)$ is non-negative, a necessary and sufficient condition that $\lim_{\mu=0}\int_a^b f(\mu, x)dx = \int_a^b f(x)dx$ is (a): that the absolute continuity of the indefinite integral $\int f(\mu, x)dx$ be uniform for all μ(de la Vallée Poussin). Another well known condition which is sufficient, being more restrictive than (a), is (b): that a summable function $g(x)$ exist such that $f(\mu, x)<g(x)$ for all μ. In this paper we find a necessary condition, bearing upon $f(x)$ itself, in order that a function $f(\mu, x)$ belonging to a certain class and satisfying (a) may also satisfy (b). The class of approximations considered includes the Poisson integral and also the Landau and Weierstrass integrals as special cases.

48. Dr. Caroline E. Seely: *Kernels of positive type.*

The author shows that if a kernel $K(s, t)$ is such that an infinite number of its iterated kernels are of positive type with respect to all linear combinations of its principal functions, then the resolvent kernel has all its poles real and simple.

R. G. D. RICHARDSON,
Secretary

THE FIFTY-THIRD REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-third regular meeting of the San Francisco Section of the Society was held at the University of California on Saturday, October 29, 1927. The total attendance was forty-four, including the following twenty-three members of the Society:

Alderton, Bernstein, Biggerstaff, Blichfeldt, Buck, Cajori, Corbin, Edwards, M. W. Haskell, E. R. Hedrick, Hotelling, Frank Irwin, R. L. Jackson, Vern James, Langford, D. N. Lehmer, S. H. Levy, McCarty, F. R. Morris, Noble, Pauline Sperry, Morgan Ward, A. R. Williams.

In the absence of the regular Chairman, Professor Allardice, Professor Blichfeldt acted as Chairman in the morning session and Professor Cajori in the afternoon session.

The following officers were elected for the coming year: Chairman, E. T. Bell; Secretary, B. A. Bernstein; Program Committee, F. L. Griffin, W. A. Manning, J. H. McDonald, B. A. Bernstein (ex officio).

The date of the next meeting at the University of California was fixed as October 20, 1928.

Titles and abstracts of papers read at the meeting follow. Professor Hedrick's address was delivered at the request of the Program Committee. Professor Cajori's second paper and Dr. Trjitzinski's paper were read by title. Professor Uspensky was introduced by Professor Blichfeldt, and Miss Weiss by Professor Manning.

1. Professor E. R. Hedrick: *Extensions of the theory of functions of a complex variable.*

The traditional theory of functions of a complex variable is limited by assumptions originally made by men often not conscious of the restrictions they imply. One extension of the traditional theory is made by acceptance of the Dirichlet definition of a function. While this creates a theory coextensive with the theory of point transformations of a plane, many ideas of the function theory are extensible: among those mentioned are Riemann surfaces, the Tissot indicatrix, pseudo-Stieltes integrals and

integration theorems, extensions of the Morera and other tests for analytic functions. A well known extension of the traditional theory is the theory of functions of several complex variables, treated briefly by several writers, and very extensively by Osgood in the second volume of his *Funktionentheorie*. Certain facts not foreshadowed by the traditional theory are mentioned. Efforts toward extension in other directions have led to the development of hypercomplex numbers on the one hand, and to extensive developments in the field of differential equations on the other hand. Many results in Differential Geometry, particularly those on conformal transformations, and those in the theory of surfaces associated with Beltrami's equations, may be regarded properly as direct extensions of the traditional theory. Finally, the attempts to extend the theory to three dimensions are reviewed, including recent papers by the author and others.

2. Professor H. Bateman: *Transverse seismic waves on the surface of a semi-infinite solid composed of heterogeneous material.*

This paper will appear in full in an early issue of this Bulletin.

3. Professor E. T. Bell: *On the arithmetic of abelian functions.*

The arithmetic applications of the abelian functions appear to be distinguished from those of the elliptic by this cardinal difference: the abelian functions in general yield properties of *simultaneous* diophantine equations, while the elliptic only exceptionally refer to such. An extremely fertile field for new properties of simultaneous diophantine systems is presented by the singular abelian functions (those whose periods are connected by one or more singular relations in the usual sense) and by the classic biquadratic relations between the thetas of more than one variable. The present note defines a new type of arithmetic invariance relevant to *simultaneous* representations in a set of forms in the *same* indeterminates, and illustrates this invariance for certain systems of quadratic forms in six indeterminates deduced from the singular abelian functions in two variables, the invariant of the singular relation having the value five.

4. Professor Florian Cajori: *Robert Burton's horoscope and the year of his birth.*

Whether Robert Burton was born in 1576, 1577 or 1578, the year of birth on his horoscope being illegible, was determined, for want of regular ephemeris, by comparing the positions of Jupiter and Saturn in his horoscope with those in other horoscopes. The year was found to be 1577.

5. Professor Florian Cajori: *Comparison of methods of determining calender dates by finger reckoning.*

In this paper Professor Cajori compares the procedures in determining calendar dates by use of the fingers, found in an anonymous writer of the

fourteenth century, in Anianus, in C. Fernandez Tranquoso, and in B. Francisco de Ossorio.

6. Dr. Vern James: *Primitive linear homogeneous groups of variety two or three.*

The variety of a group is defined and this definition is shown to be consistent with the definition of the variety of an abelian group. Theorem 2 of the paper proves that there are no primitive groups of variety two in more than two variables. Theorem 3 proves that the order of a primitive group of variety three is not divisible by a prime number greater than 7. By two other theorems the order of a primitive group of variety three is reduced to the form 2^a3^b when the number of variables exceeds four.

7. Professor C. H. Langford: *Some features of logical and mathematical propositions.*

Everyone recognizes that the propositions of logic and mathematics differ in certain important respects from propositions which occur in the natural sciences; but there is no very general agreement concerning the precise nature of these differences. This paper considers some views which are current concerning the nature, in certain respects, of the propositions of logic and mathematics; and it is especially concerned with an analysis of propositions which would commonly be said to be necessary propositions, and with an analysis of relations of entailing, incompatibility, and the like, as they occur among propositions and properties. Necessary and sufficient conditions for the truth of propositions of the kinds "p entails q" and "p is incompatible with q" are suggested.

8. Professor W. J. Trjitzinski: *Behavior at infinity of functions determined by their initial values.*

In a paper to appear in the Annals of Mathematics the author gave a representation of functions of a real variable which are determined by their initial values and in particular include quasi-analytic functions, in the form $\sum a_i f(m_i x)$, where m_i and $f(x)$ satisfy certain conditions. With additional restrictions on $f(x)$ the limit of a function so represented exists for x tending to infinity along the real axis. An explicit expression is available which enables us to study at infinity the classes of functions in question.

9. Professor J. V. Uspensky: *On a problem concerning the approximate evaluation of definite integrals.*

The author discusses the evaluation of certain definite integrals by means of continued fractions. Conditions on the integral are obtained to make the continued fraction regular and periodic.

10. Mr. Morgan Ward: *The general theory of recurring series.* Preliminary report.

The theory of linear difference equations of the nth order with coefficients in an arbitrary Galois field and elements matrices over the field

is developed with examples by postulational methods, and the structure and periods of the associated cycles of marks are determined. Connections are established with the Galois field theory, diophantine analysis and the theory of ideals.

11. Mr. Morgan Ward: *On the theory of ideal numbers.* Preliminary report.

A theory of ideal numbers is developed and applied to cubic, quartic and certain cyclotomic fields. Its connection with Zolotareff's and Dedekind's theories and its relation to results in other branches of algebraic arithmetic is given.

12. Miss Marie J. Weiss: *Primitive groups which contain substitutions of prime order p and of degree $6p$ or $7p$.*

In this paper the author proves that the degree of aprimitive permutation group (not alternating or symmetric) which contains a permutation of prime order $p(>7)$ and of degree $6p(7p)$ does not exceed $6p+6(7p+8)$.

B. A. BERNSTEIN,
Secretary of the Section

THE TWENTIETH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The twentieth regular meeting of the Southwestern Section was held at Washington University on Saturday, November 26, 1927. The total attendance was thirty-five including the following twenty-two members:

Nola Anderson, Herman Betz, Florence Black, Brenke, Davisson, Dunkel, Gerst, Gouwens, Byron Ingold, Louis Ingold, Harry Levy, G. O. James, Jaeger, Osborn, Pennell, Rider, Edna Robinson, Roever, Eugene Stephens, Stouffer, J. S. Turner, Wyant.

Professor Roever occupied the chair being relieved during the sessions by Professor Brenke. The morning session was devoted to the reading of the papers listed below. During the afternoon session Professor E. B. Stouffer gave a special lecture by invitation of the program committee on *Some canonical forms and associated canonical expansions in projective differential geometry.*

The titles and abstracts of the papers read are given below. The papers of Dr. Ayres and Professor Whyburn were read by title.

1. Professor W. C. Brenke: *Polynomial solutions of a class of difference equations.*

The author considers linear difference equations of the second order, of the form $y_n=(\alpha_n+\beta_n x)y_{n-1}-\gamma_n y_{n-2}$ where the coefficients α_n, β_n, γ_n are independent of the parameter x. Solutions of this equation which are polynomials in x, when such exist, are obtained in explicit form as special cases of the secular determinant. Many well known polynomials may thus be written as determinants.

2. Professor W. H. Roever: *Some geometric relations concerning the theory of the Eötvös torsion balance.* Preliminary communication.

In the weight field of force of the earth the lines of force have a first curvature which differs slightly from zero, and the level surfaces have, at each point, principal curvatures which differ slightly from each other. Small though these differences be, they determine those horizontal components of the weight which constitute the moments that operate the Eötvös torsion balance. By means of certain geometric theorems, which are proved in this paper, it is possible to determine directly, without any analytic treatment, expressions for the moments which operate the balance, and also to detect a relation which is usually overlooked in the theory of the balance.

3. Professor Florence Black: *A reduced system of differential equations for the invariants of ternary forms.*

A homogeneous function of the coefficients of a ternary form of order n is invariant under the general linear transformation if and only if it is a solution of a certain complete system of nine linear partial differential equations of the first order. By the introduction of a set of new variables, F. Junker has reduced the number of differential equations in the complete system to seven. A comparison with the study made by E. B. Stouffer on invariants of linear homogeneous differential equations suggests a different choice of new variables. The author of the present paper has found that the introduction of these new variables serves to reduce the number of differential equations in the complete system to six. Moreover the resulting equations are far simpler in form than those obtained by Junker's reduction.

4. Professor Louis Ingold: *Surfaces in five-dimensional space.*

In this paper a vector function $f(u, v)$, depending on two scalar parameters, is considered. It is assumed that all third partial derivatives of

the vector f are linearly expressible in terms of the five first and second partial derivatives. It is shown how the coefficients in these partial differential equations for the vector f may be expressed in terms of certain fundamental quantities which include the E, F, G of ordinary differential geometry.

5. Professor G. O. James: *Remarks on euclidean approximations to Einsteinian dynamics.*

This paper contains a tentative examination of the assumptions made in approximating the Einsteinian motion of a particle in the solar field by a potential function in euclidean space.

6. Mr. W. O. Pennell: *Operational systems.*

This paper outlines several classes of operators which follow the fundamental laws of algebra, namely, the distributive, associative, and commutative laws and the index laws. The definition of the operators includes the interpretation of fractional exponents, as well as integral exponents. Multiplication and division is an operational process, and an example is given of algebraic division performed by an operational method. The name "gamma operator" is suggested for the operator whose definition equation is $p^{\alpha}x^{n}=\Gamma(n+1)^{1/2}x^{n-\alpha}/\Gamma(n-\alpha+1)^{1/2}$. This operator parallels, in most respects, the operation of differentiation and integration; it also defines fractional integration and differentiation. The important properties of this operator are briefly outlined and discussed.

7. Dr. W. L. Ayres: *On continuous curves having certain properties.*

In this paper it is proved: (1) if every arc of a plane continuous curve M is a subset of a simple closed curve (an open curve) [a ray] of M then M is the entire plane or M is a simple closed curve (every point of M is a cut-point) [every point of M except possibly one is a cut-point]; (2) the boundary B of a plane domain D is uniformly regularly accessible from D with respect to the bounded continua of B if and only if (a) every component of B is a point or a simple continuous curve, (b) if J is any simple closed curve and $\epsilon>0$ is any number, then there are but a finite number of components of B of diameter $>\epsilon$ that have points interior to J; (3) a continuum containing no cut-point is a simple closed curve if every two points of it may be separated by two points. A necessary and sufficient condition in order that a continuous curve contain an open curve is given.

8. Professor G. T. Whyburn: *Concerning Menger regular curves.*

The following are among the results proved in this paper. (1) If no maximal cyclic curve of a continuous curve M contains an infinite collection of mutually exclusive simple closed curves, then (a) M is a Menger regular curve which is a *Baum im kleinen* at all save a countable number of points, (b) the ramification points of M are countable, and (c) every

connected subset of M is arcwise connected. (2) A cyclically connected continuous curve is a *Baum im kleinen* if and only if it has only a finite number of ramification points. (3) The ramification points of a continuous curve M are countable if and only if the ramification points of each maximal cyclic curve of M are countable. (4) If the set of all ramification points of a continuous curve M is punctiform, then M is a Menger regular curve. (5) If a bounded continuum M is disconnected by the omission of any one of its countably infinite subsets, then M is a *Baum im kleinen*. (6) A continuous curve M is disconnected by the omission of any two of its points lying together on some simple closed curve in M if and only if every maximal cyclic curve of M is a simple closed curve.

9. Professor G. T. Whyburn; *Accessibility theorems.*

In this paper the author proves the following results. (1) If the limit point P of a point set R is not regularly accessible from R, then there exists a positive number ϵ and an infinite sequence of points of R which has P as a limit point and such that no two of these points can be joined in R by any arc of diameter $<\epsilon$. (2) Every one dimensional (Menger-Urysohn sense) point P of a continuum M in $n(n\geq 3)$ dimensions is regularly accessible from each complementary domain of M whose boundary contains P. (3) In order that every point of a non-dense continuous curve M in a plane S should be regularly accessible from $S-M$ it is necessary and sufficient that M should contain no infinite collection of mutually exclusive simple closed curves. (4) If M is a plane non-dense continuous curve, G denotes the collection of all the complementary domains of M, and T denotes the limiting set of the collection G, then T is identical with the set of all those points of M at which M is not a *Baum im kleinen*.

10. Professor J. S. Turner: *On residues of iterated powers, modulo m.*

In this paper it is proved that if f belongs to a (modulo m), and if b is prime to f, then the residues of (I) $a, a^b, a^{b^2}, \cdots, a^{b^{f-2}}$, taken modulo m, are the same in the same order, as those of (II) $a, a^2, a^3, \cdots, a^{b-1}$, provided that f is a prime, and b is a primative root of f. In all other cases (I) contains fewer distinct residues than (II). Some consequences are pointed out.

E. B. STOUFFER,
Secretary of the Section

THE VISITING LECTURESHIP OF THE AMERICAN MATHEMATICAL SOCIETY

At the annual meeting of the Society in Philadelphia last year, the Council approved the plan for creating a Visiting Lectureship of the American Mathematical Society. The primary purpose of the plan is to make available every year lectures by a distinguished mathematician at the mathematical centers which are actively interested. The financial responsibility is to rest upon the colleges and universities concerned, and is to be binding from year to year only. A Committee of the Society, consisting this year of the undersigned, is responsible for the nomination of a lecturer to the Council, for the issuing of the invitation to the Visiting Lecturer after he has been appointed by the Council, and for the oversight of the general arrangements.

By means of this plan it is hoped that knowledge of recent and important mathematical advances can be brought more quickly to the various centers, and that the opportunity for informal contacts will prove of scientific value.

The support of the plan obtained for this year has been notable. Professor Constantin Carathéodory of the University of Munich has accepted the invitation to become the first Visiting Lecturer. He is expected to arrive in America at the beginning of 1928, and to remain through the following summer. He will be on the staff at Harvard University during the entire second semester of this academic year, and on the staff of the University of California during the Summer Session at Berkeley during the summer of 1928. His schedule as Visiting Lecturer is being arranged so as to avoid conflicts with these duties. A schedule of lectures during the month of January, 1928, appears on page 122 of this number of this Bulletin.

G. D. BIRKHOFF, *Chairman*, G. A. BLISS, E. R. HEDRICK.

MATHEMATICAL RIGOR, PAST AND PRESENT*

BY JAMES PIERPONT

1. *Introduction.* The Mengenlehre of Cantor, or the theory of aggregates (sets), has brought to light a number of paradoxes or antinomies which have profoundly disturbed the mathematical community for a quarter of a century. Mathematical reasoning which seemed quite sound has led to distressing contradictions. As long as one of these is unexplained in a final and conclusive manner there is no guarantee that other forms of reasoning now in good standing may not lead to other contradictions as yet unsuspected. For ages the reasoning employed in mathematics has been regarded as a model of logical perfection; mathematicians have prided themselves that their science is the one science so irrefutably established that never in its long history has it had to take a backward step.

No wonder then, that these paradoxes of Burali-Forti (1897), Russell, and others produced consternation in the camp of the mathematicians; no wonder that the foundations on which mathematics rest are being scrutinized as never before. Elaborate attempts are now in progress to give mathematics a foundation as secure as it was thought to have in the days of Euclid or of Weierstrass. Personally we do not believe that absolute rigor will ever be attained and if a time arrives when this is thought to be the case, it will be a sign that the race of mathematicians has declined. However, the aim of this paper is not to show this, but rather to pass in review some typical examples of what were regarded at the time as good mathematical demonstrations, somewhat as a

* Presented by invitation of the Program Committee at the Annual Meeting of the Society, held in connection with the meetings of the American Association for the Advancement of Science, at Nashville, December 28, 1927.

historical pageant presents to our eyes famous persons in chronological sequence.

2. *The English School of the Eighteenth Century.* Let us start with the eighteenth century, a century in which the efforts of mathematicians were largely spent in perfecting the calculus and in applying it to geometry, astronomy, and the natural sciences. The founders, Newton and Leibnitz, have left no clear account of its principles, and those who took up the new science promptly became embroiled in labyrinthine disputes. We who regard the infinitesimal calculus as nothing more than a calculus of limits can for the most part read into the obscure and scanty statements of Newton and Leibnitz what they perhaps wished to say. But their contemporaries and immediate followers had no clear idea of limits as we have, moreover their difficulties were increased by the fact that neither Newton* nor Leibnitz is quite consistent with himself.

Thus two schools arose; the English school, following Newton rested their reasoning on "the motion of bodies accelerated according to various hypotheses" and on "prime and ultimate ratios"; the continental school, following Leibnitz, rested their reasoning on infinitely small, non-archimedian quantities or "little zeros." Let us exhibit a few examples from the two schools; we shall take first the English.

A highly esteemed work was the *Treatise on Fluxions*, by Thomas Simpson (1st edition, 1737; 2d edition, 1776). As is well known, the proof that Newton gave in his *Principia* that $(uv)'=uv'+vu'$ is not satisfactory. This apparently was recognized by Simpson, who gives an alternate proof resting on the derivative of $y=x^2$.

On page 1 we read: "All kinds of magnitudes are to be considered as generated by the continual motion of some of their bounds or extremes; as a line by the motion of a point;

* As for Newton, see an illuminating article by A. De Morgan, Philosophical Magazine, (4), vol. 4 (1852), p. 321.

a surface by the motion of a line; and a solid by the motion of a surface. Every quantity so generated is called a variable or flowing quantity; and the magnitude by which any flowing quantity would be uniformly increased in a given portion of time with the generating celerity at any proposed position or instant (was it from thence to continue invariable) is the fluxion of the said quantity at that position or instant."

Thus in our language the fluxion of a variable u at the time t is $(du/dt)\Delta t$. Simpson represents this by $\dot{u}$. Let us see now how Simpson gets the fluxion of $y=x^2$. On a straight, let the reader mark points A, r, R, B in order, and on another straight the points C, s, e, S, D. Simpson reasons textually as follows: "Conceive two points m, n to proceed at the same time from two points A, C, along the right lines AB and CD in such sort that $CS=y$ is always equal to the square of $AR=x$, which latter moves uniformly. Furthermore let r, s, R, S, be any contemporary positions of the generating points. If $rR=v$ we have $CS=y=x^2$, $Cs=(x-v)^2=x^2-2xv+v^2$ and hence $Ss=CS-Cs=2xv-v^2$. From whence we gather that while the point m moves over the distance v, the point n moves over the distance $2xv-v^2$. But this last distance (since the square of any quantity is known to increase faster in proportion than the root) is not described with an uniform motion (like the former) but with an accelerated one. It therefore is equal to, and may be taken to express, the uniform space that might be described with the mean celerity at some intermediate point e in the same time. Therefore, seeing the distances that might be described in equal times, with the uniform celerity of m and the mean celerity at e are as v to $2xv-v^2$, or as $\dot{x}$ to $2x\dot{x}-v\dot{x}$, it is evident that in the same time the point m would move uniformly over the distance $\dot{x}$ the other point n with its celerity at e would move uniformly over the distance $2x\dot{x}-v\dot{x}$. This being the case let r, R and s, S be now supposed to coincide, by the arrival of the generating points at R and S, then e (being always between s and S) will likewise coincide with S; and the distance $2x\dot{x}-v\dot{x}$ which

might be uniformly described in the aforesaid time with the velocity at e (now at S) will become barely equal $2x\dot{x}$ which (by the definition) is equal to $\dot{y}$, the true fluxion of Cn or x^2."

To avoid the difficulty in Newton's proof, Simpson has introduced the point e, which merely replaces one stumbling block by another. To realize the great difference between this English school and the modern doctrine of limits, one has only to compare this bungling proof with the simple little proof in any calculus of today.

3. *The Continental School of the Eighteenth Century.* We now turn to the continental school founded by Leibnitz. Perhaps the first systematic presentation of Leibnitz's methods to be published was the *Analyse des Infiniment Petits*, by the Marquis de l'Hospital (1696), which enjoyed the most widespread popularity. An English translation *The Method of Fluxions*, by E. Stone, appeared in 1730. De l'Hospital's book was largely founded on a little treatise by John Bernoulli, *Die Differentialrechnung*,* written in 1691–92, but only published in 1922 on the occasion of the tercentenary celebration of the Bernoulli family in Basel.

The treatise begins on page 11 with three postulates, of which the first reads: "A quantity which is diminished or increased by an infinitely small quantity is neither increased nor decreased." On page 12, we find: "The differential of x^2 is $2x\ dx$, which is proved thus: $(x+e)^2=x^2+2ex+e^2$, subtracting x^2 gives $2ex+e^2$ as remainder, and this on account of postulate 1 is $2ex=2xdx$.

"The differential of x/y is $(y\,dx-x\,dy)/y^2$. For if we subtract x/y from $(x+e)/(y+f)$ we get $(ey-fx)/(y^2+fy)=$ by postulate 1, $(ey-fx)/y^2=(ydx-xdy)/y^2$."

The next writer of this school whom we wish to consider is the immortal Euler, one of the greatest figures in the whole history of mathematics. Let us look at his *Institutiones*

* We quote from *Die Differentialrechnung von Johann Bernoulli*, Ostwald's Klassiker, Nr. 211.

Calculi Differentialis (1775). In the preface he considers the differential of $y=x^2$. He gives x the increment ω, the corresponding increment of y is $2x\omega+\omega^2=\eta$; the ratio η to ω is $2x+\omega$ to 1. Hence this ratio approaches $2x$ the smaller ω is taken. He is thus led to define the differential calculus as the method of determining the ratio of evanescent increments. These evanescent quantities are called differentials "que, cum quantitate destituantur, infinite parva quoque dicuntur, quae igitur sua natura ita sunt interpretanda, ut omnino nulla seu nihilo aequalia reputentur." He admonishes the reader to bear in mind that these differentials are absolutely zero and that nothing can be inferred from them other than their mutual ratio, which is in the end reduced to a finite quantity (verum perpetuo tenendum est, cum haec differentialia absoluta sint nihila, ex iis nihil aliud concludi nisi eorum rationes mutual, quae utique ad quantitates finitas deducuntur).

Thus Euler accepted unqualifiedly the notion that there exist quantities which are absolutely zero, yet whose ratios are finite numbers. The reader who wishes further information regarding Euler's views may consult Chapter III of the above work, entitled *De infinitis atque infinite parvis.* He encourages the reader here by remarking that this notion does not hide so great a mystery as is commonly thought, and which in the mind of many renders the calculus suspect. Any doubts which may have arisen will be shown devoid of foundation as the theory is developed. This reminds one of d'Alembert's statement: "Allez en avant, la foi vous viendra."

Euler's *Differential and Integral Calculus* and his *Introduction to Infinitesimal Analysis* were the standard textbooks of the day; they were in everybody's hands. The fame of the author and their great popularity renders it necessary to give another example of his style of reasoning.

How does he find the differential of $y=\log x$? This is treated in §180 of the *Institutiones*. Replacing x by $x+dx$ gives

$$dy = \log(x + dx) - \log x = \log(1 + dx/x).$$

Now in Chapter VII of volume I of the *Introduction to Analysis* he has found

$$(1) \qquad \log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots.$$

Replacing here z by dx/x gives

$$dy = dx/x - dx^2/(2x^2) + dx^3/(3x^3) - \cdots.$$

As all the terms of this series beyond the first are evanescent we have

$$d \cdot \log x = dx/x.$$

We turn thus to the *Introductio in Analysin Infinitorum* (1748) to learn how the series (1) is established. We find (§115, seq.) the demonstration rests on Newton's celebrated binomial formula

$$(2) \qquad (1+u)^m = 1 + mu + \frac{m(m-1)}{1 \cdot 2}u^2 + \cdots.$$

Euler gives no proof of this, which in those days was proved in algebra.* Let us see, however, how he uses (2) to prove (1). Euler starts with the relation $a^\omega = 1 + k\omega$, ω infinitely small; then a being taken as base, $\omega = \log(1+k\omega)$. Hence

$$a^{i\omega} = (1 + k\omega)^i = 1 + \frac{i}{1} \cdot k\omega + \frac{i(i-1)}{1 \cdot 2}k^2\omega^2 + \cdots.$$

Set $i = z/\omega$, z finite; then $\omega i = z$, and

$$a^z = 1 + kz + \frac{(i-1)}{2i}k^2z^2 + \frac{(i-1)(i-2)}{2i \cdot 3i}k^3z^3 + \cdots.$$

As i is a "number larger than any assignable quantity"

$$\frac{i-1}{2i} = \frac{1}{2}, \quad \frac{(i-1)(i-2)}{2i \cdot 3i} = \frac{1}{2 \cdot 3}, \text{ etc.},$$

hence

---

* Judged by modern standards, these demonstrations are quite worthless.

$$(3) \qquad a^z = 1 + kz + k^2z^2/2! + k^3z^3/3! + \cdots .$$

The larger i is taken, the nearer $(1+k\omega)^i$ is to 1. Euler thus sets

$$(1 + k\omega)^i = 1 + x, \quad \therefore k\omega = (1 + x)^{1/i} - 1,$$
$$i\omega = i[(1 + x)^{1/i} - 1]/k.$$

Hence

$$\log (1 + x) = i(1 + x)^{1/i}/k - i/k.$$

Now

$$(1 + x)^{1/i} = 1 + \frac{x}{i} - \frac{(i - 1)}{2i^2}x^2 + \cdots ;$$

hence if i is infinitely large

$$i(1 + x)^{1/i} = i + x - x^2/2 + \cdots ,$$

and thus

$$(4) \qquad \log (1 + x) = [x - x^2/2 + x^3/3 - \cdots]/k.$$

Setting $z=1$ in (3) gives

$$a = 1 + k + k^2/2! + k^3/3! + \cdots .$$

As a has been left arbitrary, we may take it so that $k=1$; calling this value e, we have

$$e = 1 + 1 + 1/2! + 1/3! + \cdots ,$$

while (4) goes over into the desired formula (1).

This demonstration from a Weierstrassian standpoint is about as bad as it could be; but then, are we not told now by the intuitionalists that a large part of the Weierstrassian mathematics is devoid of proof, if indeed it is not nonsense? Let us therefore be charitable. We have taken space to give this proof because it is entirely typical.

We have not space to follow the further history of "the little zeros" of Leibnitz and Euler. As Americans, it may interest us sufficiently to note that our greatest mathematician of earlier days, B. Peirce, used them without hesitation in

his remarkable treatise *Curves, Functions, and Forces* (2 vols., 1841), as may be seen in Book II, Chapter 2, page 172 of volume 1.

4. *The Method of Lagrange.* Another eddy in the current of mathematical thought was produced by Lagrange's *Théorie des Fonctions Analytiques* (1st edition, 1797; 2d edition, 1813). He is dissatisfied with the little zeros of Leibnitz, Bernoulli, and Euler "which although correct in reality are not sufficiently clear to serve as foundation of a science whose certitude should rest on its own evidence." He is as little satisfied with the fluxions of Newton, which introduces a foreign notion, that of motion; moreover "one can see by the learned *Treatise on Fluxions* by Maclaurin how difficult it is to demonstrate the method of fluxions and how many ingenious artifices we must employ to demonstrate the different parts of this method."

Lagrange therefore proposes to get rid of little zeros and of limits at one stroke by founding the calculus on the development of a function in a power series (Taylor's or Maclaurin's). With becoming modesty he remarks that it is strange that this method of establishing the calculus did not occur to mathematicians earlier, especially to Newton, the inventor of the method of series and of fluxions. Let us see what this grand idea is.

If in $f(x)$ we replace x by $x+h$, "it becomes $f(x+h)$ and by the theory of series one can develop it in a series

$$(1) \quad f(x+h) = f(x) + ph + qh^2 + rh^3 + sh^4 + \cdots."$$

"Not to make gratuitous assumptions," Lagrange begins by examining the form of the series (1), and shows that the exponents of h are in fact integers and not fractions. He next studies the coefficients $p, q, r, \cdots$. To this end he observes that if we replace h by $h+k$ in (1) we get

$$(2) \quad \begin{cases} f(x+h+k) = f(x) + (h+k)p + (h+k)^2q + \cdots \\ \qquad\qquad = f(x) + hp + h^2q + h^3r + \cdots \\ \qquad\qquad\quad + kp + 2hkq + 3h^2kr + \cdots \text{ etc.} \end{cases}$$

Also, if we replace x by $x+k$ in (1), we get

$$(3)\quad f(x+h+k) = f(x+k) + hp(x+k) + h^2q(x+k) + \cdots .$$

Now

$$f(x+k) = f(x) + kf_1(x) + \cdots ,$$
$$p(x+k) = p(x) + kp_1(x) + \cdots ,$$
$$q(x+k) = q(x) + kq_1(x) + \cdots , \text{ etc.}$$

These in (3) give

$$(4)\quad \begin{aligned} f(x+h+k) = {} & f(x) + hp + h^2q + h^3r + \cdots \\ & + kf_1(x) + hkp_1(x) + h^2kq_1(x) + \cdots . \end{aligned}$$

Comparing (2) and (4) gives

$$(5)\qquad p = f_1(x), \quad q = \tfrac{1}{2}p_1(x), \quad r = \tfrac{1}{3}q_1(x), \cdots .$$

Lagrange now observes that we get q from p in the same way that we get p from $f(x)$, and a similar remark holds for the other coefficients r, s, $\cdots$ in (1).

He calls $f_1(x)$ the first derivative of $f(x)$; it is merely the first coefficient in the development (1), and not a differential coefficient obtained by a passage to the limit. He denotes it by $f'(x)$. This function also has a first derivative which he denotes by $f''(x)$ etc. Thus the relations (5) give

$$p = f'(x), \quad q = \frac{1}{2!}f''(x), \quad r = \frac{1}{3!}f'''(x), \cdots ,$$

which in (1) give

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2!}f''(x) + \cdots .$$

Lagrange now observes: "This new expression has the advantage of showing how the terms of the series depend on each other, and especially how when one knows how to form the first derivative function, one can form all the derivative

functions which enter the series." Then a little later he adds: "For one who knows the rudiments of the (traditional) differential calculus, it is clear that these derivative functions coincide with

$$dy/dx, \quad d^2y/dx^2, \cdots."$$

This then is the grand scheme: no more infinitesimals, no more prime and ultimate ratios whose principles have given rise to endless disputes. Differential coefficients are merely the coefficients in the development of a function.

When a modern reader looks over reasoning like this and bears in mind that Lagrange was one of the greatest mathematicians of all time, he is amazed. The great gulf that separates mathematical reasoning of to-day from that of date 1813 is brought home very clearly to him. The consoling feature about the work that we have seen thus far is this: the results are right although the reasoning is faulty. The intuition of these great men is far in advance of their logic. Is it not quite likely that a similar state of affairs holds in the theory of aggregates today?

Lagrange's method of development of the calculus free from the knotty questions regarding infinitesimals and limits was received with considerable favor. It suffers however a mortal defect. It rests upon the assumption that a given function can be developed in a power series, and there is no known method of deciding this question independently of the thing he wishes to avoid, namely limits.

5. *The Standpoint of Cauchy.* With Euler and Lagrange, a well defined period in the history of mathematics is closed. Euler is the last great formalist. We have just seen that Lagrange is fully alive to the objectionable reasoning of many of his contemporaries, and we have witnessed his vain efforts to give the calculus "toute la rigueur des démonstrations des Anciens," as he says. The true method was not Newton's, nor Euler's, nor Lagrange's, and yet it lay close at hand and had already been clearly stated by

d'Alembert in the *Encyclopédie*,* in his article *Différentiel*. Here he writes: "What we most need to treat here is the metaphysics of the differential calculus. This metaphysics about which one has written so much is more important and perhaps more difficult to develop than even the rules of this calculus." He remarks that certain mathematicians cannot admit the suppositions which are made concerning infinitesimals, as they are false in principle and capable of leading to wrong results. He then goes on to say: "But when one observes that all the facts that have been discovered by ordinary geometry may be established by the calculus much more easily, one cannot help concluding that this calculus furnishes certain simple and exact methods, and rests on principles just as simple and certain."

These he thinks form the metaphysics of Newton, which Newton has revealed only in part. D'Alembert reads into Newton what Newton ought to have said but which (if we are not mistaken) no one had seen before; and this we believe is d'Alembert's great discovery. He says: "Newton has never regarded the differential calculus as a calculus of infinitesimals, but as a method of prime and ultimate ratios, that is to say, a method of finding the limit of these ratios. Newton has never differentiated quantities, but only equations, since every equation embraces a relation between two variables, and the differentiation of equations is merely finding the limit of the ratios between finite differences of the two variables which the equation involves."

In another article, on *Limits*, he says: "The theory of limits is the true metaphysics of the differential calculus."

The first treatise to adopt this standpoint to the exclusion of any other was (as far as I have ascertained) the *Traité Élémentaire* of Lacroix (2d edition, 1806; I have not seen the first edition). It is, however, with Cauchy that the new

* *Encyclopédie ou Dictionnaire Raisonné des Sciences*, vol. 10, Geneva, 1770. This is, I believe, a pirated copy of the original French edition. I have not seen this latter; probably d'Alembert's article on the calculus was published for the first time some years earlier.

era begins. In his *Cours d'Analyse* (1821), his *Résumé des Leçons sur le Calcul Infinitésimal* (1823), his *Leçons sur le Calcul Différentiel* (1829), and in the *Leçons de Calcul Différentiel et de Calcul Intégral* (2 vols., 1840), by his pupil L'Abbé Moigno, we behold for the first time the foundations of the calculus developed with a rigor which is near to that of our own today. De Moigno in his *Introduction* observes: "Before Cauchy published his treatises, the demonstrations of the fundamental theorems of the calculus rested too often on the consideration of certain series which were used without discernment, without having examined their convergence, or if indeed they represented the functions which gave them birth. It was a veritable abuse against which Cauchy never ceased to protest. Never did he employ the development of a function in a series without first establishing its possibility, its form, its convergence, in a word its right to represent the given function." This we think is rather exaggerated, but it is true on the whole.

If one asks what is one of Cauchy's most obvious oversights as to rigor, we would say that it is overlooking the care that one must take in a very common process in analysis, i.e., the interchange in the order of passing to the limit in a double limit. Thus in his *Cours d'Analyse* (p. 131), he states that $F(x)=\sum u_n(x)$ is continuous if F is convergent and the u_n are continuous. Under the same conditions (*Résumé des Leçons*, Œuvres (2), vol. 4, p. 237),

$$\int_a^b F\,dx = \sum \int_a^b u_n\,dx.$$

Also (ibid., p. 195)

$$\frac{\partial}{\partial u}\int_a^b f(x,u)\,dx = \int_a^b \frac{\partial f}{\partial u}\,dx,$$

etc.

The justification of these and similar interchanges of limits rests on the notion of uniform convergence, which was not discovered till a later date (Stokes, 1847; Seidel, 1848; Cauchy, 1853).

Cauchy's standard of rigor was immeasurably in advance of his contemporaries (we except Gauss of course); it served as model for a generation of mathematicians. Abel,* writing to Holmboe from Paris (1826), says of Cauchy: "He is at present the only one who knows how mathematics should be treated." In another letter,† speaking of Taylor's development, he says: "I have found only a single rigorous demonstration, that of M. Cauchy."

6. *The Standards of Weierstrass.* However, even after making various minor improvements relative to double limits, etc., the last word on rigor had not been said; there was a still higher standard for which to strive. Indeed, so vastly superior was this new standard in the minds of some of its proselytes that they looked on the rest of their fellow mathematicians as living in utter darkness.

What was the new doctrine and who was its founder? Both questions can be answered by one word, it is a name revered and honored the world around, Weierstrass.

What the Weierstrassian doctrine is, is too well known to you for me to dwell upon. I may be allowed, however, to mention one or two matters which will come up for discussion when we reach the next era, the era of to-day. The great step in advance that Weierstrass took was to arithmetize analysis. Before then, many analytical facts were accepted as self evident. For example: if $f(x)$ is continuous in (a, b) and has opposite signs at a and b, then $f(x)=0$ at some point within the interval. One has merely to picture the graph of this function to see intuitively the truth of this theorem. That this type of reasoning is inadmissible rests on the following observation. Let us define with Cauchy: The function $f(x)$ is continuous for $x=c$ if $\lim f(x)=f(c)$ for $x\to c$. One assumes now that such continuous functions are coextensive with the class of functions as pictured by our geometric intuition. Many examples show us the contrary.

* Œuvres, vol. 2, p. 250.

† Ibid., p. 257

Perhaps the one that startled the easy going world the most was Weierstrass' continuous function having at no point a derivative. Another hardly less remarkable example was Peano's curve which passes through every point of a square. Finally we note a fundamental problem: What is the meaning of the limit of a sequence $a_1, a_2, a_3, \cdots$?

To arithmetize analysis one had to ban geometric reasoning; all proof must rest entirely on pure analysis. This forced Weierstrass to find an arithmetic equivalent of our geometric notion of an interval or segment of a straight line, i.e., to lay down an arithmetic theory of irrational numbers. Only then was he able to prove arithmetically the simplest properties of continuous functions, properties which had been accepted previously as too obvious to require even a passing remark. To meet the requirements of the new standards of rigor a great deal of the reasoning which had been regarded as valid had to be supplemented, or when too bad, to be replaced by considerations of a different type. If one wishes to form an idea how great these changes were, one cannot do better than to compare attentively the first edition of C. Jordan's *Cours d'Analyse* (1882–1887) with the second (1893–1896). In the calculus of variations the devastation produced by the teaching of Weierstrass was even greater, and the work of repair far more difficult.

By the end of the last century the Weierstrassian standards of rigor had won over practically the whole world. It was believed that absolute rigor had been reached, that on the foundations of Weierstrass our mathematical edifice, whose giddy summits seem to reach the clouds, rests so securely that nothing can disturb its massive repose. Illustrious names can be cited to support this statement; one will suffice. In his address before the Paris Congress (1900), Poincaré,* reviewing the arithmetization of mathematics, asks: "Have

* Compte Rendu du Deuxième Congrès International des Mathématiciens, p. 115.

we at last attained absolute rigor? At each stage of its evolution our forerunners believed they too had attained it. If they were deceived, are we not deceived like them?" After discussing the question he gives his verdict: "One may say to-day that absolute rigor has been attained." Alas, to-day, a scarce quarter of a century after these memorable words were uttered some there are who think we have been living in a fool's paradise. The mighty edifice is tottering, they think, perhaps soon to fall leaving a horrible ruin in its place. I personally do not believe this to be the case and this view is shared by most of the mathematicians of to-day. And yet when one hears one of the greatest living mathematicians calmly telling the world that a considerable part of our analysis is devoid of proof, if it is not nonsense, and when one beholds the mighty efforts which the champions of Weierstrass are making to repel these attacks, it is only reasonable, in view of such facts, to ask ourselves, "Is all well?"

We shall use the remainder of this address to give an account of this last stage in the evolution of mathematical rigor. It goes back to Kronecker, who confessed one day to Netto* that he had spent far more time thinking in philosophy than he had in mathematics. Philosophy and mathematics are not good companions.

If I had entered more fully into the early history of our theme, you would doubtless have been bored by a great deal of talk about its metaphysics. When the mathematician has not been clear in his own mind he has had recourse in the past to metaphysics. Bertrand Russell once humorously defined philosophy as the science that makes simple things complicated. Now the mathematician trained in the school of Weierstrass was fond of referring to his science as the absolutely clear science. Any attempt to drag in metaphysical speculation was resented with indignant energy. How the times have changed! O tempora, o mores! Now see

* E. Netto, Mathematical Papers, International Mathematical Congress, Chicago, 1893.

what philosophy did to Kronecker; later we shall see what it has done to Brouwer and Weyl, and what a revolution it wishes to bring about.

7. *Kronecker's Position.* To make Kronecker's position clear we may illustrate it by an example taken from algebra. Every one knows how fundamental the notion of reducibility is. Before Kronecker, algebraicists were content to say a polynomial $f(x)$ might have a rational factor, in which case $f(x)$ is reducible; in the contrary case it is irreducible. This is merely a logical definition of these terms; it is a case of "either," "or" with no means in sight as far as the definition is concerned to decide whether the given $f(x)$ is reducible or is not.

In his great Festschrift *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, Journal für Mathematik, vol. 92 (1882), Kronecker announces for the first time the new doctrine: "The definition of reducibility given in §1 is devoid of a sure foundation until a method is given by means of which it can be decided whether a given function is irreducible or not by the definition." In a footnote, he calls attention to the fact that "a similar need (often indeed unnoticed) is manifest in many other cases, in definition as well as in proof, and I shall take this up on another occasion in a general and careful manner." Only to a very minor degree did Kronecker carry out his intention, and that only in the field of algebra and algebraic numbers. In the great field of analysis he did nothing except to criticize in lectures and conversation the work of his contemporaries. Let us see how deep the new program cuts. The foundation of exact analysis is the real number system, whether defined as by Weierstrass, or as by Dedekind, or as by Méray and Cantor. Of two real numbers a and b, either $a=b$ or $a>b$ or $a<b$, and definitions are given for each case. According to Kronecker they are only definitions in appearance, since these definitions do not give the means of deciding in each case which of these three alternatives holds.

Another example is the fundamental theorem of Bolzano-Weierstrass relative to the upper and lower limits of a limited function. The well known proof consists in establishing the logical existence of a sequence of intervals one within the other, but it does not in general give the means of calculating these limits. The definition, therefore, is not a valid definition and must be rejected, from Kronecker's standpoint.

In the same year (1882) that Kronecker promulgated his new doctrine, Lindemann succeeded in showing that π is not algebraic, and so showed the futility of trying to square the circle. In a conversation with Lindemann,* Kronecker asks "Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are nonexistent?" Kronecker's attitude is made still clearer by the following extract of a letter of Weierstrass† to Mme. Kowalevski (1885): "I say that a so-called irrational number has as real an existence as any other object of the mind; Kronecker on the contrary regards it now as an axiom that only equations between whole numbers exist . . . .

"It makes the matter worse that Kronecker uses his authority to support the view that all who have labored on the foundations of the function theory are sinners before the Lord. When a person like Christoffel says that in twenty or thirty years the present function theory will be buried, and that analysis will be reduced to a theory of forms, one answers with a shrug of the shoulders; but when Kronecker makes this assertion which I reproduce word for word: 'If I still have the time and the energy, I will myself show the mathematical world that not only geometry but also arithmetic can point the path to analysis, and certainly a more rigorous one. If I cannot do this, then another will who comes after me, and the world will recognize the inexactitude of the types of proof now employed in analysis,' such a statement I say,

* H. Poincaré, *Wissenschaft und Hypothese*, 3d edition, 1914, p. 252.

† G. Mittag-Leffler, *Une page de la vie de Weierstrass*, Paris Congress (1900), pp. 150–151.

. . . is not only humiliating to those who are thus recommended to acknowledge their error and abjure what has been the object of their unremitting thoughts and efforts, but it is also a direct invitation to the younger generation to leave their present leaders and to group themselves about him as the apostle of a new doctrine which, it is true, has yet to be established. Really this is sad and fills me with bitter pain." The revolutionary movement inaugurated by Kronecker in 1882 apparently died an early death from sheer inanition. It was easy for him dogmatically to assert: "Definitions must contain the means of reaching a decision in a finite number of steps and existence proofs must be conducted so that the quantity in question can be calculated with any required degree of accuracy," but outside of algebra he took no steps to realize his program, nor did any one for a quarter of a century seriously make the attempt. We therefore leave this movement for the present and consider briefly quite another subject.

8. *Rigor in Geometry.* So far we have said nothing of rigor in geometry. The reason is obvious; more than two thousand years before, geometry went through the process which we have just sketched for analysis. In the *Elements* of Euclid we have the results of a long period of critical analysis of the most acute Greek minds. After the downfall of the antique world, the intellectual world of Western Europe lay buried in darkness till the great awakening in the sixteenth and seventeenth centuries. No wonder that the achievements of Greek learning seemed almost superhuman to a people who had outgrown the futile subtleties of scholasticism. To the mathematicians of the Renaissance and long after, the *Elements* of Euclid was a work of superlative excellence. Let the following serve as an example.* It is "a work whose propositions have such an admirable connection and dependence that take away but one, and the whole falls; whose method is the most just, admitting and advancing nothing

* Benjamin Martin, *Biographia Philosophica*, London, 1764, p. 56.

without a demonstration, and no demonstration but from what foregoes, and these so convincing, elegant, and perspicuous that it is beyond the skill of man to contrive better. Here the most artful and diligent carpers have never been able to set a footing."

However the "artful and diligent carpers" have found first one "footing" and then another, so that to-day it is generally recognized that the *Elements* are far, very far, from being a perfect work. Definitions are given which do not define, axioms are implied but not stated, certain proofs are needlessly complicated. Perhaps the most unsatisfactory feature relates to congruence and the implication of motion or displacement. On the other hand there is much that we to-day admire not with the blind idolatry of the past but with the critical knowledge of a connoisseur. We have a great advantage over the Greeks therein that a study of various non-euclidean geometries has enabled us to realize the difficulties which beset an entirely rigorous treatment of euclidean geometry.

There is no space in this address to discuss the history of the critical movement in geometry; we must however mention one aspect of it on account of its great importance in the thought of to-day relative to the foundations of analysis. We refer to the axiomatic treatment of the foundations of geometry which reached its highest excellence in Hilbert's *Grundlagen der Geometrie* (1899). Instead of trying to define a point, a straight, a plane, this method introduces three sets of things and subjects them to certain relations called axioms. One has no mental picture of these things; the reasoning makes no call on our geometric intuition, it is purely formal. On this account it is absolutely necessary to give a proof of freedom of contradiction and this is done by showing that any contradiction would involve a contradiction in arithmetic.

9. *Arithmetic.* The non-contradiction of the axioms of geometry is thus referred to the axioms of our number

system. Are the laws of arithmetic non-contradictory? However far we carry our developments, can we ever arrive at a result $1=0$, for example? No one in his senses has ever believed this; yet mathematics is not a science which rests on faith, but on proof. We are thus led back to our main theme, the foundations of analysis. A most important part of this theory relates to the number system.

We may begin our discussion by asking "What are the integers 1, 2, 3, $\cdots$?" The question is as elusive as the question, "What is a straight line?" We do not expect the philosophers to agree as to what a number is, and indeed they have not agreed; but we do expect the mathematicians to be unanimous. Do not numbers form the very basis of all analysis? E. G. Husserl in his *Philosophie der Arithmetik* (1891) observes "Not one significant question do I know which those concerned have answered with even tolerable harmony."

The first attempt to establish rigorously the laws of arithmetic was, as far as I know, by R. Dedekind in *Was sind und was sollen die Zahlen* (1887). He bases his development on infinite aggregates, i.e., as he defines them, on sets which can be put in uniform correspondence with a subset. It was incumbent on him to prove the existence of such sets. His reasoning (page 17) rests on the set of all things and is thus open to the objection of Burali-Forti's paradox.

G. Frege's *Grundgesetze der Arithmetik* (vol. 1, 1893; vol. 2, 1903) also makes use of the aggregate theory. At the close of vol. 2 (p. 253) he remarks "A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr. Bertrand Russell as the work was nearly through the press." In his despair Frege's only comfort is "Solatium misereris, socios habuisse malorum," if indeed this is a comfort. His criticism of his predecessors was merciless; what a bitter pill it must have been for him to illustrate the old adage, "Errare humanum est!" Russell's criticism has to do with the paradox named after

him, namely, the set of all sets which are not members of themselves.

These unsuccessful attempts of Dedekind and Frege to found the number system on the notion of infinite aggregates have brought us now face to face with the notorious contradictions in Cantor's Mengenlehre which we referred to at the beginning of this address. Paradoxes were no new thing in philosophy. Did not Epimenides, the Cretan, say that all Cretans are liars, and did not Zeno the Eleate show that Achilles could not overtake the tortoise? They were something very new, however, in mathematics. Burali-Forti's paradox (1897) was like a bomb, soon to be followed by other bombs of like nature. Consternation spread among mathematicians; what was to be done? The greatest authorities were called in consultation; their diagnoses were poles apart: quot homines, tot sententiae. Poincaré* at the Rome Congress (1908) went so far as to say "Later generations will regard the Mengenlehre as a disease from which one has recovered." To get rid of these paradoxes, various lines of action have been adopted. One may do as Cantor, Dedekind and Frege did: give up, quit the field. For such we would recall a saying of DeMorgan† prompted by the contradictions arising from divergent series: "The history of algebra shows us that nothing is more unsound than the rejection of any method which naturally arises on account of one or more apparently valid cases in which such method leads to erroneous results. Such cases should indeed teach caution, but not rejection; if the latter had been preferred to the former, negative quantities, and still more their square roots, would have been an effectual bar to the progress of algebra · · · and those immense fields of analysis over which even the rejectors of divergent series now range without fear, would have been not so much as discovered, much less cultivated and settled."

* O. Hölder, *Die mathematische Methode*, Berlin, 1924, p. 556.

† A. DeMorgan, *Differential and Integral Calculus*, London, 1842, p. 566.

Other lines of action rest on the answer to the query: when is a mathematical object or process defined? Some say, its definition can employ but a finite number of words. For example Poincaré, discussing Zermelo's well ordering of the continuum said:* "There are two cases; either one asserts that the method of well ordering is expressible in finite terms ('endlich aussagbar'), in which case the assertion is not proved; · · · or we allow the possibility that the method is not 'endlich aussagbar'? In this case I can attach no meaning to the procedure; for me it is only empty words."

In another place,† Poincaré asks: "Is it possible to reason on objects which cannot be defined by a finite number of words? Is it even possible to speak of them, knowing what one is speaking of, pronouncing only empty words? Or on the contrary should one not regard them as unthinkable? For my part, I do not hesitate to respond that they are pure nonentities." Mathematicians of this stamp are called finitists; their position is hotly attacked by the Cantorians.

Poincaré holds that the paradoxes of the theory of aggregates are due to the use of non-predicative definitions. In his *Science et Méthode* (Paris, 1912), page 207, he says: "Thus the definitions which should be regarded as non-predicative are those which contain a vicious circle." On page 208 he declares: "A definition which contains a vicious circle defines nothing." This view is also held by B. Russell. The Cantorians claim that this position cannot be maintained, since without such definitions modern analysis would be robbed of some of its most valuable results.‡

10. *The Logistic Group.* Without making too fine distinctions we may say that those engaged in laying the new foundations of analysis fall into three groups:

---

* H. Poincaré, *Sechs Vorträge*, Leipzig, 1910, p. 48; they were held at Göttingen in 1909.

† *Dernières Pénsées*, Paris, 1913, p. 132.

‡ E. Zermelo, *Neuer Beweis für de Möglichkeit einer Wohlordnung*, Mathematische Annalen, vol. 65 (1907), p. 107.

(1) The logistic: the Italian school is led by Peano, the English by Russell and Whitehead.

(2) The axiomatic: led by Hilbert.

(3) The intuitionist: led by Brouwer.

The first point to note about the logistic group is the fact that their work is written in a sign-language invented by Peano and extended by Russell. Relatively few people are willing to learn this language and these works therefore are for the most part unknown. We may summarize some of their leading ideas as follows. Their cardinal idea is that mathematics is a part of logic. The concepts and processes of mathematics on careful analysis are found to be few in number and to admit a symbolic treatment analogous to algebraic manipulations. By giving each symbol a precise unique meaning one hopes to avoid pitfalls that beset ordinary mathematical reasoning due to the ambiguity of common language. Moreover the symbolism itself is an aid in reasoning in the same way that algebraic symbols help us in ordinary arithmetic. The logistic program is to reduce all mathematics to symbolic logic; this has already been partly accomplished.* One of the great difficulties in this program is to avoid the contradictions and paradoxes of the theory of aggregates. Russell observed that these contradictions could all be avoided by using his vicious circle principle.† "Whatever involves all of a collection must not be one of the collection." To make this observation effective he has invented a theory of types. In this theory propositional functions and classes form a hierarchy according to their possible arguments, also a distinction is made between the various functions belonging to the same argu-

* Formulaire de Mathématique, vol. 1, 1895; vol. 2, 1897–99; vol. 3, 1901; vol. 4, 1902–03. The different parts are written by a variety of persons, Peano, Burali-Forti, Vailati, Padoa, Vivanti, etc. See also A. N. Whitehead and B. Russell, *Principia Mathematica* (3 vols.), 2d edition, 1925–27.

† B. Russell, American Journal of Mathematics, vol. 30 (1908), p. 225. Whitehead and Russell, *Principia Mathematica* (2d edition, 1925), p. 58.

ment. By this means all known contradictions of the aggregate theory are avoided, and mathematical induction is established. In order to establish other parts of analysis, for example the irrational numbers, Russell is obliged to introduce a certain axiom, the axiom of reducibility. This axiom states that any combination or disjunction of predicates is equivalent to a single predicate. By this means the order of a nonpredicative function can be lowered by one, so that after a finite number of steps we reach an equivalent predicative function. This axiom of reducibility seems to have excited universal opposition. One author states that its introduction is an act of harikari. Ramsey* holds that there is no reason to suppose that it is true, and if it were, it would be a happy accident and not a logical necessity. It has no place in mathematics, and what cannot be proved without it cannot be regarded as proved at all. Ramsey believes he has discovered how the work of Wittgenstein† can be utilized so as to free the *Principia* from the objections which have caused its rejection by the majority of German authorities.

11. *The Axiomatic Group.* The great leader of this group is Hilbert. His masterly treatment of the foundations of geometry already referred to quite prepared the mathematical public to lend a willing ear to his address *Ueber die Grundlagen der Logik und der Arithmetik* at the Heidelberg Congress, 1904. After referring to the difficulties which beset a rigorous development of the number system due in part to paradoxes of the aggregate theory, Hilbert announces his program as follows: "I believe that all the difficulties which I have touched upon may be overcome and an entirely satisfactory foundation of the number concept can be reached by a method which I call the axiomatic method, and whose leading idea I wish now to develop."

* F. P. Ramsey, *The foundations of mathematics*, Proceedings of the London Mathematical Society, (2), vol. 25 (1926), p. 338.

† L. Wittgenstein, *Tractatus Logico-Philosophicus*, London, 1922.

Hilbert has treated this subject in five other papers, and I do not believe he has said his last word yet. These papers are:

Ueber den Zahlbegriff, Jahresbericht der deutschen Mathematiker Vereinigung, vol. 8 (1900).

Axiomatisches Denken, Mathematische Annalen, vol. 78 (1918), p. 405.

Neubegründung der Mathematik, Abhandlungen des Mathematischen Seminars der Hamburgischen Universität, vol. 1 (1922), p. 157.

Die Logischen Grundlagen der Mathematik, Mathematische Annalen, vol. 88 (1922–23), p. 151.

Ueber das Unendliche, Mathematische Annalen, vol. 95 (1926), p. 161.

As in the days of Newton and Leibniz, so now the notion of infinity is our greatest friend; it is also the greatest enemy of our peace of mind. We may compare it to a great waterway bearing the traffic of the world, a waterway however which from time to time breaks its bounds and spreads devastation along its banks. Weierstrass taught us to believe we had at last thoroughly tamed and domesticated this unruly element. Such however is not the case; it has broken loose again. Hilbert and Brouwer have set out to tame it once more. For how long? We wonder.

Hilbert thinks it can become our useful servant and preserve all its powers uncurtailed. Brouwer thinks this is impossible; we can at most build a canal through our territories and allow a fraction of the infinite to pass through it.

We use the notion infinity in two ways illustrated by the integers 1, 2, 3, 4, · · · . In one sense, this is an endless sequence, such that after each element there follows another; in the other, we regard them in their totality, as a finished product. We have two other common infinite notions, space and time; neither has a bound. After each moment of time there is another; after each point on a straight there is another. We never think of these things as closed or finished.

Now Gauss wished* only such a conception of the infinite in any part of mathematics. For him there was no actual infinity, only an endless growing or becoming. It is this notion of an actual infinity which lies at the basis of Cantor's theory of transfinite numbers. Of this theory Hilbert says: "This seems to me the most admirable fruit of the mathematical mind and in fact one of the highest achievements of man's purely intellectual processes." After remarking on the catastrophic effects that the paradoxes of Russell etc. have caused, he declares: "No one shall drive us out of the paradise that Cantor created for us." We have seen what Poincaré thought of this paradise: that it is a non-entity. We will presently examine Brouwer's ideas about it.

Hilbert tells us that for a quarter of a century the questions relating to the foundations of mathematics have never been out of his thoughts, yet in his last paper *Ueber das Unendliche*, he is forced to admit that the present state of affairs relative to the paradoxes of the aggregate theory is intolerable. However, let us be comforted, for Hilbert assures us: "There is an entirely satisfactory way to escape the paradoxes without betraying our science." This goal can be reached only when the notion of the infinite has been made entirely clear.

To this end he turns to the physical sciences. He finds that the tendency in modern science is an emancipation from the infinite. Matter is not continuous, but atomic; so also is energy. The infinite space of our forefathers has shrunk to a finite volume, and has become elliptic.

How does this accord with our mathematical concepts? Does not nature show us that we are on the wrong track in dealing with the infinite? In opposition to Frege and Dedekind, who thought to develop the number system independently of intuition or experience by employing an actual infinity, Hilbert finds that a prerequisite of scientific knowledge is a fund of intuitive ideas; pure logic is not

* Werke, vol. 8, p. 216.

sufficient. Operations on the infinite can be rendered certain by an appeal to the finite. In Kant's philosophy, an idea is a concept of the intellect which transcends all experience, and by which the concrete of our senses is made complete as a totality. This is the role left to infinity: it is an idea.

Although Hilbert is quite opposed to Russell and Whitehead in the belief that mathematics is a part of logic, it is interesting to note how the methods of symbolic logic creep into Hilbert's work. Kepler found the conic sections of the ancients ready at hand. Einstein found the tensor analysis of Ricci and Levi-Civita likewise. Behold the same pre-established harmony is again made manifest. The symbolic logic of Peano, Russell, and Whitehead lies before him like a ripe fruit ready to be picked.

Hilbert takes this logic but reinterprets it. For him it is a sign-language which puts mathematical theorems into formulas and which expresses logical reasoning by formal processes. Signs and symbols of operation are freed from their significance with respect to content. The axioms of mathematics merely express the rules by which formulas follow one another. Beyond these meaningless signs and formulas which constitute mathematics there is a "meta-mathematics" which deals only with the concrete and never employs but a finite number of logical steps of a kind universally admitted.

The first axioms laid down therefore relate to the finite; for them the laws of ordinary logic hold; their freedom of contradiction is intuitive. We must however pass to the transfinite and use "all," "there is," "the excluded middle," "complete induction" etc. To this end Hilbert introduces the axiom

$$A(\tau A) \rightarrow A(a)$$

which he says means in ordinary language "If a predicate A applies to the object τA, it applies to all objects a." To illustrate this Hilbert supposes A is the predicate "is venal." Then $A(\tau A)$ is to be regarded as a definite person of such

uncorruptible sense of justice that should it turn out that he were venal, it would follow that all mankind is venal.

Hilbert regards this transfinite axiom as adjoined to the finite axioms just as in algebra we adjoin to the real number system the imaginary numbers, and in geometry we adjoin to real space the ideal plane at infinity. Still more striking he finds the analogy with the ideals in the theory of algebraic numbers. How strange it comes about, Hilbert exclaims, that this (transfinite) form of reasoning, which Kronecker so passionately opposed, proves to be the exact analog of Kummer's ideals, these numbers which Kronecker so ardently admired and praised as the highest mathematical achievement.

Having found a system of axioms of sufficient generality for all the needs of modern analysis, there remains the final and most important step: the placing of the keystone to the arch, which in this case is the proof of the freedom from contradiction of these axioms. Having outlined how this is to be accomplished, Hilbert illustrates the power of his methods of proof by disposing (rather summarily) of such oft-debated problems as complete induction, the Weierstrass theorem of the existence of upper and lower limits, Zermelo's axiom.

12. *Intuitionalism.* The chief figure in this group is L. E. J. Brouwer, professor of mathematics at the University of Amsterdam. He early attained international eminence for his extraordinarily subtle and far reaching papers in the theory of point sets (analysis situs). His doctor's dissertation (1907) was *Over de grondslagen der wiskunde* (*On the foundation of mathematics*). His inaugural address as professor at the Amsterdam university (1912) was on *Intuitionism and formalism.* A translation of this by Professor A. Dresden appeared in this Bulletin, vol. 20 (1913–14). Beginning with 1918, Brouwer has expounded his theory in a series of papers in the Proceedings of the Amsterdam Academy of Sciences, in the Mathematische Annalen, and in the Journal für Mathematik.

It appears that Brouwer, like Kronecker and H. Weyl, is very much of a philosopher; his view of mathematics we would expect to be much influenced by his philosophical speculations. If he holds certain mathematical tenets radically different from the usual ones of to-day, his justification will probably be of a philosophical nature.* We recall that Kronecker declared that if he did not live to carry out his program of overhauling mathematics from the bottom up, another would come who would. The fulfillment of this prophecy seems to be Brouwer.

Brouwer not only agrees with Kronecker that a definition must contain the means of constructing the object defined, but he carries this idea to its logical end and declares the use of the logical form known as the *tertium non datur*, or excluded middle, is legitimate only for finite sets. Brouwer believes† this law arose from the consideration of finite sets in mathematics; after being adopted by logic it was given an a priori existence independent of its mathematical origin, and then by virtue of its a priori character it was unjustly extended by mathematicians to infinite sets. By adopting these two restrictions Brouwer robs himself of two of the most powerful aids of research in modern analysis. Now no one would have an objection to any mathematician's trying to see what he could do under certain restrictions; one might regret such efforts as a waste of time, or look at the affair as a sporting proposition, like swimming the English Channel with one's hands tied. This is not Brouwer's attitude, however. Like Kronecker, he does not hesitate to tell his contemporaries that they are wrong. Thus in his *Intuitionistische Mengenlehre*,‡ he says that the Komprehensionsaxiom, by virtue of which all things which have a certain property are united to form an aggregate (Menge), even in

* The reader may consult a paper by A. Dresden, *Brouwer's contribution to the foundations of mathematics*, this Bulletin, vol. 30 (1924), p. 31.

† Jahresbericht der Vereinigung, vol. 28 (1919), p. 203.

‡ Proceedings, Amsterdam Academy, vol. 23 (1922), p. 949.

the limited form used by Zermelo, is inadmissible in founding the aggregate theory, or is at least unserviceable. Only in a constructive definition of aggregates is a trustworthy foundation of mathematics to be found. Further the principle of the excluded middle is an unpermissible means of proof, which can be allowed only a scholastic or heuristic value; so that theorems which cannot be proved otherwise are devoid of any mathematical content. There is no sufficient ground for admitting this principle; logic rests on mathematics, and not conversely. In his paper* *Ueber die Bedeutung des Satzes vom ausgeschlossen Dritten*, Brouwer remarks: "On this foundation, particularly in the last half century, extensive false theories have been erected. The contradictions which have been repeatedly encountered have brought to life the formalistic criticism." He grants that their axiomatic treatment will avoid contradictions, "but nothing of mathematical value will be attained in this manner; a false theory which is not halted by a contradiction is none the less false, just as a criminal policy unhindered by a reprimanding court is none the less criminal." These are strong words, and strong words are usually met by others. This is Hilbert's counterblast.† "What Weyl and Brouwer are doing, is mainly following in the path of Kronecker; they are trying to establish mathematics by throwing overboard everything which does not suit them and dictatorially promulgating an embargo. The effect of this is to dismember and cripple our science and to run the risk of losing a large part of our most valuable possessions. Weyl and Brouwer condemn the general notions of irrational numbers, of functions, even number theoretic functions, Cantor numbers of higher classes etc., the theorem that an infinite set of positive integers has a least, and even the 'tertium non datur', so for example the statement: either there is only a finite number of primes or there are infinitely

* Journal für Mathematik, vol. 154 (1925), p. 1.

† *Neubegründung der Mathematik*, Abhandlungen des Mathematischen Seminars der Hamburgischen Universität, vol. 1 (1922), p. 157.

many. These are examples of forbidden theorems and modes of reasoning. I believe that powerless as Kronecker was to abolish irrational numbers (Weyl and Brouwer do allow us to retain a torso), no less powerless will their efforts prove to-day. No, Brouwer's program does not signify a revolution, but merely the repetition of a vain coup de main with former means, but which then was undertaken with greater dash, yet totally failed. To-day the State is thoroughly armed and strengthened through the labors of Frege, Dedekind, and Cantor. Weyl and Brouwer's efforts are doomed in advance to futility."

We have stated the two main theses of Brouwer: the rejection of the comprehension axiom as used by Cantor and Zermelo, and the law of the excluded middle.

In general the rejection of the law of the excluded middle produces great complication, which we are inclined to believe represents an element of strength rather than of weakness in Brouwer's theory. To go further into Brouwer's theory would require fresh definitions, and space does not permit this.

All new theories have to struggle for existence and recognition. Brouwer's theory is not easy to read. So far his theory has not made matters simpler; quite the reverse. It may be that this is necessary. Cauchy's methods forced the reasoning of his predecessors into innocuous desuetude; Weierstrass showed Cauchy's reasoning was far from perfect. Is Brouwer destined to lay down the standards of the next generation?

YALE UNIVERSITY

A FURTHER NOTE ON THE CONVERSE OF FERMAT'S THEOREM

BY D. H. LEHMER

In a previous paper* the writer had discussed the converse of Fermat's theorem as a means of establishing the primality or non-primality of a large integer. Use was made chiefly of the following theorem:

THEOREM 3. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$ *and if* $a^x \equiv r \not\equiv 1$ *for* $x = (N-1)/p$ *and if* $r-1$ *is prime to* N, *then all the factors of* N *belong to the form* $np^\alpha+1$ *where* α *is the highest power of the prime* p *contained in* $N-1$.

It is the purpose of this note to give a more general theorem in which the third part of the hypothesis of Theorem 3 is removed.

THEOREM 4. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$ *and* $a^x \equiv r \not\equiv 1$ *for* $x = (N-1)/p$, *then all the factors of* N/δ *are of the form* $np^\alpha+1$, *where* α *is the highest power of the prime* p *contained in* $N-1$ *and where* δ *is the G.C.D. of* $r-1$ *and* N.

Let k be a prime factor of N/δ and let ω be the exponent to which a belongs modulo k. Then ω divides $N-1$ and $k-1$ but not $m = (N-1)/p$; for if ω divided m we would have $a^m \equiv 1 \pmod{k}$ so that $r-1$ would divide by k. But this is impossible, since k divides N/δ which is prime to $r-1$. From here on, the proof is the same as in Theorem 3 with the result that $k = np^\alpha+1$.

Ordinarily, we have $\delta = 1$ so that the two theorems become identical. An example in which this is not the case is the following: Let $N = 16{,}046{,}641$. $N-1 = 2^4 \times 3^3 \times 5 \times 17 \times 19 \times 23$. It will be found that

* This Bulletin, vol. 33 (1927), pp. 327–340.

$$2^{N-1} \equiv 1 \pmod{N},$$
$$2^{(N-1)/19} = 8708025 = r.$$

If we take the G.C.D. of $r-1$ and N, we get $\delta = 35113$, so that $N/\delta = 457$ and we are able to say that the factors of 457 are of the form $19n+1$. The factors of δ may be obtained by applying Theorem 4 in which a new value of m is chosen. The modulus may be taken as δ.

Theorem 4 has been applied to the number

$$N = (2^{61} + 1)/3 = 768614336404564651.$$

This number has been listed as a prime but the writer could not find any account of its investigation.*

The factorization of $N-1$ is

$$N - 1 = 2(2^{60}-1)/3 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 41 \cdot 61 \cdot 151 \cdot 331 \cdot 1321.$$

It was found that

$$3^{(N-1)/5} \equiv 754529885435533861 = r_1,$$
$$3^{(N-1)/3} \equiv 243065045915817725 = r_2,$$
$$3^{N-1} \equiv 1. \pmod{N}.$$

It happens that r_1-1 and r_2-1 are both prime to N. Hence the factors of N belong to the forms

$$\left.\begin{matrix} 3n+1 \\ 25n+1 \\ 61n+1 \end{matrix}\right\} 4575n+1.$$

By making use of the fact that N has no factor $<300{,}000$ and by seeking to represent N as the difference of two squares it was easily shown to be a prime.

Two more numbers dividing $10^n \pm 1$ have also been tested recently. The first of these is $(10^{37}-1)/9$ or

$$N = 1{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111.$$

* Cunningham and Woodall, *Factorisation of* $y^n \pm 1$, London, 1925, p. 1.

It was found that

$$7^{N-1} \equiv 618,117,398,624,349,204,361,513,620,865,505,749 \pmod{N}.$$

Hence N is composite. This number furnishes another example of the scarcity of primes of this form. The next such number which has any chance of primality consists of 47 of the digits 1.

The second number tested is $(10^{41}+1)/11$ or

$$N = 9,090,909,090,909,090,909,090,909,090,909,090,909,091.$$

In this case it was found that

$$3^{N-1} \equiv 763,287,007,500,473,474,161,903,784,495,157,879,509 \pmod{N}.$$

It follows, then, that N is also composite. This result represents the sixth attempt and failure to discover a larger prime than $2^{127}-1$ found by Lucas in 1877.

THE UNIVERSITY OF CALIFORNIA

ON THE APPROXIMATE REPRESENTATION OF ANALYTIC FUNCTIONS*

BY DUNHAM JACKSON

The purpose of this paper is to discuss the convergence of approximating polynomials determined by a least-square criterion, together with certain auxiliary conditions. Let $f(x)$ be a given function over the interval $a \leqq x \leqq b$. For each positive integral value of n, let p_n be a positive integer $\leqq n$. A polynomial of the nth degree may be required, for example, to coincide in value with $f(x)$ at p_n specified points of the interval; and among the infinitely many polynomials of the

* Presented to the Society, September 8, 1927.

nth degree satisfying this requirement, an approximating polynomial may be determined by the condition that the integral of the square of the error shall be a minimum.* Or it may be required that the polynomial and its first p_n-1 derivatives shall agree in value with $f(x)$ and its first p_n-1 derivatives at a single specified point. Or, more generally, contact of more or less high order may be prescribed at each of a number of different points, in such a way that the total number of auxiliary conditions adds up to p_n.

In carrying through the proof of convergence, it will be supposed that $f(x)$ is analytic over a certain region of the complex plane containing the interval (a, b) of the real axis in its interior. Some of the preliminaries, however, can be dealt with just as easily under less restrictive hypotheses.

The existence of a minimum in each of the cases to be considered is an almost immediate corollary of well known existence proofs for the problem without auxiliary conditions. For the main point of the proof there is to show that if an upper bound is assigned for the integral of the square of the error (with polynomials of given degree) the coefficients must belong to a bounded region, which may be taken as closed,† and the imposition of the new conditions merely restricts the choice of the coefficients to a closed manifold in this region. And the uniqueness of the minimizing polynomial is proved by the usual argument that if two different polynomials give equally good approximations their average gives a better one.‡ So these questions may be dismissed without further consideration.

---

* For the case of trigonometric approximation, a problem of this character with auxiliary conditions independent of n has been treated in a recent paper by the writer: *Note on a problem in approximation with auxiliary conditions*, this Bulletin, vol. 32 (1926), pp. 259–262. The method there was quite different, the highly specialized character of the auxiliary conditions being compensated by greater generality in the hypotheses on $f(x)$.

† See, for example, D. Jackson, *On functions of closest approximation*, Transactions of this Society, vol. 22 (1921), pp. 117–128; pp. 118–121.

‡ See Transactions, loc. cit., pp. 121–122.

To proceed with a closer characterization of the approximating polynomials to be studied, let $f(x)$ be a given real function, defined for $a \leqq x \leqq b$, and n a given positive integer. For simplicity of statement, let it be supposed that $f(x)$ has $n+1$ continuous derivatives throughout (a, b). Then, if $P(x)$ is any polynomial, and if $P(x_1) = f(x_1)$ for a value of x_1 in the interval, the difference $f(x) - P(x)$, expressed by means of Taylor's theorem with the remainder, is either of the form $(x-x_1)^\nu \psi(x)$, where $1 \leqq \nu \leqq n$ and $\psi(x)$ is continuous and different from zero for $x = x_1$, or else it is of the form $(x-x_1)^{n+1}\psi(x)$, where $\psi(x)$ is bounded. In the former case the root is of multiplicity ν, and in the latter case, while not necessarily of determinate multiplicity, it may be said to be of multiplicity $\geqq n+1$.

Let $x_1, x_2, \cdots, x_q$ be q distinct points of the interval (a, b), where q is a positive integer $\leqq n$. With each of these points x_k, let a positive integer ν_k be associated, so that $\nu_1 + \nu_2 + \cdots + \nu_q = p \leqq n$. (If in particular $q = n$, of course each ν_k must be equal to 1.) Let $P_n(x)$ be a polynomial of the nth degree with real coefficients such that, for each x_k,

$$(1) \qquad P_n(x_k) = f(x_k), \quad P_n'(x_k) = f'(x_k), \cdots,$$
$$P_n^{(\nu-1)}(x_k) = f^{(\nu-1)}(x_k) \qquad (\nu = \nu_k).$$

There would be one and just one polynomial of degree $p-1$ satisfying these conditions,* and since $p-1 < n$, they are satisfied by infinitely many polynomials of the nth degree. In particular, let $P_n(x)$ be determined among all such polynomials of the nth degree by the requirement that

$$\int_a^b [f(x) - P_n(x)]^2 dx$$

shall be a minimum. *Then $f(x) - P_n(x)$ must be orthogonal to every polynomial $\pi_n(x)$ of the nth degree which vanishes with its first $\nu_k - 1$ derivatives at each of the points x_k.* For if there were a polynomial $\pi_n(x)$ satisfying these conditions, and not ortho-

* See, for example, Markoff, *Differenzenrechnung*, Leipzig, 1896, pp. 1–3.

gonal to $f(x)-P_n(x)$, the polynomial $P_n(x)+h\pi_n(x)$ would satisfy the auxiliary conditions imposed on $P_n(x)$, for arbitrary h; the integral

$$\int_a^b [f(x) - P_n(x) - h\pi_n(x)]^2\,dx,$$

which satisfies the conditions for differentiating with regard to h under the sign of integration, would have a derivative reducing for $h=0$ to

$$-2\int_a^b [f(x) - P_n(x)]\pi_n(x)\,dx \neq 0;$$

and the integral would not have a minimum for $h=0$, as it must under the hypothesis that $P_n(x)$ is the minimizing polynomial.

From this it follows further that $f(x)-P_n(x)$ *must have roots of aggregate multiplicity* $\geqq n+1$ *in* (a, b). For if this were not the case, it would be possible to construct a polynomial $Q_n(x)$, of the nth degree or lower, having roots at the same points, with exactly the same multiplicities. Since the roots of $f(x)-P_n(x)$ must include those prescribed by (1), each with at least the specified multiplicity, $Q_n(x)$ would belong to the class of polynomials denoted by $\pi_n(x)$ in the preceding paragraph. But the supposed $Q_n(x)$ is not orthogonal to $f(x)-P_n(x)$, because $[f(x)-P_n(x)]Q_n(x)$ is not identically zero, and never changes sign, having all its roots of even order. So denial of the assertion in italics results in a contradiction.

The property expressed in this assertion can be made the basis of a convergence proof. Let it be supposed from now on that $f(x)$ is analytic throughout the interior and on the boundary of a circle of the complex plane with center on the axis of reals at the middle point of the interval (a, b), and with radius $R>2r$, where $r=(b-a)/2$, the function $f(x)$ being real for real values of x. Let a polynomial $P_n(x)$ be defined as above for each positive integral value of n, the numbers $q=q_n$, x_k, and ν_k being variable with n in any way, subject

to the relations of inequality that have been imposed on them. *Then $P_n(x)$ converges toward $f(x)$ throughout the interior and on the boundary of any circle of radius $\rho < R-2r$, concentric with the first.* The theorem is stated and proved in this form, in spite of the fact that the region of convergence thus obtained does not necessarily include the whole interval (a, b); it will be possible to choose ρ so as to include the entire interval, if $R > 3r$.

Let $y_1, y_2, \cdots, y_{n+1}$ be the $n+1$ roots, or $n+1$ of the roots, of $f(x)-P_n(x)$, multiple roots being indicated by repetition; it will not be necessary to specify what roots are distinct, and what roots are coincident. It is a matter of elementary algebra to see that*

$$\begin{aligned}\frac{1}{t-x} &= \frac{1}{t-y_1} + \frac{x-y_1}{t-y_1}\cdot\frac{1}{t-x} \\ &= \frac{1}{t-y_1} + \frac{x-y_1}{t-y_1}\cdot\frac{1}{t-y_2} + \frac{x-y_1}{t-y_1}\cdot\frac{x-y_2}{t-y_2}\cdot\frac{1}{t-x}. \\ &= \cdots,\end{aligned}$$

or, with the notation

$$g_k(t) = (t-y_1)(t-y_2)\cdots(t-y_k),$$

$$\frac{1}{t-x} = \frac{1}{g_1(t)} + \frac{g_1(x)}{g_2(t)} + \cdots + \frac{g_n(x)}{g_{n+1}(t)} + \frac{g_{n+1}(x)}{g_{n+1}(t)}\cdot\frac{1}{t-x}.$$

For any value of x inside the circle of radius R, by Cauchy's formula,

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-x},$$

the integral being extended around the circumference of the circle. Hence

$$f(x) = c_0 + c_1 g_1(x) + \cdots + c_n g_n(x) + R_n(x) g_{n+1}(x),$$

* For the entire convergence proof, see Runge, *Theorie und Praxis der Reihen*, Leipzig, 1904, pp. 126–142; Hermite, *Sur la formule d'interpolation de Lagrange*, Journal für Mathematik, vol. 84 (1878), pp. 70–79; pp. 70–72.

where the c's are constants, and

$$R_n(x) = \frac{1}{2\pi i} \int_C \frac{f(t)\,dt}{g_{n+1}(t)(t-x)},$$

a function analytic throughout the interior of the circle. The terms $\sum c_k g_k(x)$ constitute a polynomial of the nth degree. Since $R_n(x)\ g_{n+1}(x)$ vanishes at the points $y_1, \cdots, y_{n+1}$, the polynomial agrees in value with $f(x)$ at these points, the roots of the difference between $f(x)$ and the polynomial having (at least) the multiplicities indicated by the repetitions among the y's. But these conditions determine a polynomial of the nth degree uniquely; *the polynomial* $\sum c_k g_k(x)$ *must be identical with* $P_n(x)$. Consequently

$$f(x) = P_n(x) + \int_C \frac{(x-y_1)(x-y_2)\cdots(x-y_{n+1})}{(t-y_1)(t-y_2)\cdots(t-y_{n+1})} \cdot \frac{f(t)dt}{t-x}.$$

Now let it be understood that x is in the interior or on the boundary of the circle of radius ρ, where, as already specified, $\rho < R-2r$. Then $|x-y_k| \leqq \rho+r$, for all values of k. On the other hand, $|t-y_k| \geqq R-r$, for all values of k and for all points t on the path of integration; and $|t-x| \geqq R-\rho$. So, if M is the maximum of $|f(t)|$ on this path,

$$|f(x) - P_n(x)| \leqq \frac{2\pi RM}{R-\rho} \cdot \left(\frac{\rho+r}{R-r}\right)^{n+1}.$$

Since $(\rho+r)/(R-r) < 1$, the right-hand member approaches zero as n becomes infinite, and $f(x)-P_n(x)$ converges uniformly toward zero for all the values of x in question.

The foregoing discussion can be supplemented in various ways. The case $q_n = n+1$, which falls just outside the hypotheses, being a straightforward problem of interpolation with no place for a least-square condition, is the one treated by Runge in the passage cited. The case $q_n \leqq n$, $p = p_n = n+1$, also leaves no room for a least-square condition, but comes within the scope of the convergence proof.* In this case,

* See Hermite, loc. cit.

as well as in the preceding, it makes no essential difference if the points x_k, instead of being on the axis of reals, are anywhere in the circle of radius r. At the other extreme, the problem for $q_n=0$ is that of the Legendre series, and while the part of the discussion that relates to auxiliary conditions is irrelevant, the proof of convergence is still valid, as far as it goes; it shows, for example, that the Legendre series for an entire function converges everywhere.* In one of the most interesting special cases under the earlier hypotheses, that in which $q_n=1$, $p_n=n$ (or, a little less narrowly, $p_n=n-l$, where l is $\geqq 0$ and independent of n), the point x_1 being taken for simplicity as the middle point of the interval, Dr. D. V. Widder, with whom the writer discussed the substance of the paper in its preliminary stages, noted at once that the convergence takes place throughout the interior of the large circle of radius R; and instead of the requirement that $R>2r$ it is sufficient to assume that $R>r$. More generally, if l is independent of n, and if there are for each value of n at least $n-l$ auxiliary conditions attached to points distant by not more that r' from the middle point of (a, b), where $r'<r$, it is allowable in the convergence proof to let ρ be any number less than $R-2r'$, while the assumption with regard to R is that R is greater than the larger of the numbers r, $2r'$. To generalize in another direction, it would be possible to replace the square of the error by the mth power of its absolute value, $m>1$, and to admit a weight-function in the integral to be minimized. An exhaustive discussion of questions of this sort would be outside the scope of the present paper.

THE UNIVERSITY OF MINNESOTA

* See Whittaker and Watson, *Modern Analysis*, third edition, Cambridge University Press, 1920, pp. 322–323.

GENERALIZATIONS OF THE THEOREM OF FERMAT AND CAUCHY ON POLYGONAL NUMBERS*

BY L. E. DICKSON

1. *Introduction and Summary.* We seek the least l such that every integer $A \geqq 0$ is a sum of l values of

$$(1) \qquad p_{m+2}(x-k) = \tfrac{1}{2}(x-k)[m(x-k-1)+2]$$

for integers $x \geqq 0$. When $k=0$, (1) is a polygonal number of order $m+2$, and Fermat stated that $l=m+2$. This was first proved by Cauchy, who found that all but four of the $m+2$ polygonal numbers may be taken as 0 or 1. A simpler proof was given by the writer in this Bulletin (vol 33 (1927), pp. 719–726) that paper will be cited as I.

When all but four of the l values of (1) are 0 or 1, we shall prove here that

$l=m-1$ if $k=1$, $m \geqq 7$; $l=m-2$ if $k=2$, $m \geqq 8$, or if $k \geqq 3$, $m \geqq 7$; $l=6$ if $k=1$, $m=6$, or $k=2$, $m=7$; $l=5$ if $k=1$, $m=4$ or 5, or if $k \geqq 2$, $m=6$; $l=4$ if $k \geqq 1$, $m=3$, or if $k \geqq 2$, $m=4$ or 5.

When all but s of the l values of (1) are 0 or 1, where $5 \leqq s \leqq l$, we shall prove that, for every $k \geqq 1$,

$$l = m-2 \quad \text{if} \quad m > 7, \quad l = s \quad \text{if} \quad m \leqq 7.$$

But A is a sum of l values of (1) if and only if†

$$(2) \qquad 8mA + l(m-2)^2 = \sum_l (2mx + 2 - m - 2mk)^2,$$

summed for l integers $x \geqq 0$. We saw that $l=4$ when $k=1$, $m=3$, and when $k=2$, $m=4$ or 5. Hence (2) shows that for every integer $A \geqq 0$,

* Presented to the Society, September 9, 1927.

† This implies the like formula with k replaced by any larger integer $k+p$. The values $x \geqq p$ give the same squares as occur in (2) for $x \geqq 0$. Hence we use the least permissible k.

$$24A + 4 = \sum_4 (6x - 7)^2, \quad 40A + 36 = \sum_4 (10x - 23)^2,$$
$$8A + 4 = \sum_4 (4x - 9)^2. \tag{3}$$

By Theorem 6 and (2), we have

$$3A + 4 = \sum_4 (3x - 16)^2 \quad \text{if } A \not\equiv 4 \pmod 8. \tag{4}$$

For $s=5$, $m \leqq 7$, we saw that $l=5$. Then (2) for $k=1$ is

$$8mA + 5(m-2)^2 = \sum_5 (2mx + 2 - 3m)^2. \tag{5}$$

For $m=3$ this result is obtained by adding 1 to each member of (3_1) and is discarded. When m is 4 or 6, we cancel factors 4 or 16. We get

$$8A + 5 = \sum_5 (4x - 5)^2, \quad 40A + 45 = \sum_5 (10x - 13)^2, \tag{6}$$

$$3A + 5 = \sum_5 (3x - 4)^2, \quad 56A + 125 = \sum_5 (14x - 19)^2. \tag{7}$$

The results obtained when $s=4$, $l=5$ are consequences of these.

When $k=1$, $m \geqq 8$, Theorem 2 states that $l=m-2$ except for $A=54m+16$. This is a remarkable fact since the single exceptional value may be made as large as we please by taking m sufficiently large. Again, if $k=5$, $m=6$, we find that $l=4$ fails first for $A=980$; by increasing k, the first failure will occur for a value of A exceeding any assigned number.

In later papers I shall prove that the value of l obtained when $s=3$ is never less than the minimum l found here, and I shall give an improved method valid when $s>4$, but not when $s=4$.

2. *Formulas.* Give (1) the notation $\frac{1}{2}mx^2+\frac{1}{2}nx+c$. Thus

$$n = 2 - m - 2mk, \quad c = \tfrac{1}{2}m(k^2 + k) - k. \tag{8}$$

For $k \geqq 1$ the reduced minor conditions in I now follow from

$$A \geqq 4c + 4E, \quad A \geqq 4c + \tfrac{2}{3}m. \tag{9}$$

For, from the inequality (9_2) we deduce (11) of I, since the sum $(-n)(2m+n)/m$ of its last two terms is negative, because $-n$ is positive and $2m+n$ is negative. By (8_1), we may write (4) of I in the form

$$A = mg + 4c + b + r, \quad g = \tfrac{1}{2}(a - b - 2kb). \tag{10}$$

3. *Excess E when $k=1$.* The values of (1) for $x \geqq 0$ are $m-1$ and all the polygonal numbers of order $m+2$.

TABLE II.

SUMS BY FOUR OF $m-1$ AND POLYGONAL NUMBERS

0-4, $m-1$, m, $m+1$-5, $2m-2$, $2m-1$, $2m$, $2m+1$-6, $3m-3$, $3m-2$, $3m$, $3m+1$, 3-7, $4m-4$, $4m-1$, $4m+2$-8, $5m+1$, 2, 4, 5, 7, 8, $6m$, $6m+3$-9, $7m+3$-9, $8m+2$-10, $9m+1$, 4, 7-10, $10m+5$-11, $11m+4$-9, 11, $12m+3$, 4, 6-12, $13m+2$, 5, 7-12, $14m+6$-12, $15m+6$-9, 11-13, $16m+5$-13, $17m+4$, 5, 7-13, $18m+3$, 6, 7, 9-14, $19m+8$, 9, 11-14, $20m+7$, 10-14, $21m+7$-13, 15, $22m+6$-15, $23m+5$, 6, 8, 9, 11-15, $24m+4$, 7, 10-16, $25m+9$-13, 15, 16, $26m+8$, 10, 11, 13-16, $27m+9$, 11-16, $28m+8$-17, $29m+7$-13, 15-17, $30m+6$, 7, 9, 10, 12-17, $31m+5$, 8, 11-17, $32m+10$-18, $33m+9$, 10, 12, 13, 15-18, $34m+12$-18, $35m+11$-17, $36m+9$-19, $37m+8$-13, 15-19, $38m+7$, 8, 10, 11, 13-15, 17-19, $39m+6$, 9, 12-17, 19, $40m+11$, 12, 14-20, $41m+10$, 13, 15-20, $42m+13$-20, $43m+12$-17, 19, 20, $44m+11$-20, $45m+10$-13, 15-21, $46m+9$-21, $47m+8$, 9, 11-17, 19-21, $48m+7$, 10, 12-21, $49m+12$, 13, 15-21, $50m+11$, 14-22, $51m+13$-17, 19-22, $52m+13$-22, $53m+12$, 13, 15-21, $54m+17$-19, 21, 22.

If $m \geqq 6$, $E(9m+6) = 2$, since neither $9m+5$ nor $9m+6$ is equal to a number of Table II. If $m=5$, Table II lacks $17 = 2m+7 = 3m+2 = 4m-3$. If $m=4$, it lacks $26 = 4m+10 = 5m+6 = 6m+2 = 7m-2$. Also,* $E(54m+16) = m-5$. Hence E is not smaller than the value shown in

THEOREM 1. *For $k=1$, $E=m-5$ if $m \geqq 7$, $E=2$ if $m=6$, $E=1$ if $m=4$ or 5, $E=0$ if $m=3$.*

By (10) we require that the values of $b+r$ shall include a complete set of residues modulo m when r takes the values $0, 1, \cdots, E$, and b takes certain consecutive odd values. When $m=3$ or $m \geqq 5$, this will be true if b takes the values

* And independently of Table II, since the only partitions of 53 into 0, 1, 3, 6, 10, 15, 21, 28, 36, 45 are $1+1+6+45$, $1+1+15+36$, $1+6+10+36$, $1+3+21+28$, $0+10+15+28$, $1+10+21+21$.

$\beta, \beta+2, \beta+4$, whence $d=6$. But when $m=4$, we need only $\beta, \beta+2$, whence $d=4$. Conditions (9) hold if $A \geqq 8m$.

Let $m \geqq 7$. By (6)–(15) of I,

$$U = 24mA - 63m^2 + 12m + 36, \quad V = 2mA - m^2 + 6m + 4,$$
$$P = 7m + 2, \quad W = 3mA - 14m^2 - 2m + 4,$$
$$F = m^2A^2 - 74m^3A - 28m^2A + 193m^4 + 70m^3 - 68m^2 - 24m > 0.$$

The last evidently holds if

$$A \geqq 74m + 28. \tag{11}$$

Next, let $m=6$. Then

$$U = 36(4A - 60), \quad V = 12A - 8, \quad P = 44, \quad W = 18A - 512,$$
$$F = 36[(7A - 88)^2 - (12A - 8)(4A - 60)] = 36(A^2 - 480A + 7264),$$

and $F>0$ if $A \geqq 465$ and hence if (11) holds.

If $m \geqq 6$, Theorem 1 now follows from Lemma 3 of I and the following lemma.

LEMMA 1. *If $m \geqq 5$ in Table* II, *$E(A) \leqq m-5$ when A is between any consecutive blocks, while $E(A) \leqq 2$ when A is in any block.*

For, the difference between consecutive entries in any block is always $\leqq 3$. If r is the term free of m in the leader $qm+r$ of any block, then $r+4$ is found to be the term free of m of a number of Table II within the preceding block. Hence $qm+r-1$ is the sum of $m-5$ and the number $(q-1)m+r+4$ in the table.

When $m=5$, the last two sentences hold after we suppress from Table II all entries down to and including the last entry which differs by 3 from the next entry of the block. The only exception is the leader $9m+7$, while $9m+6$ exceeds $8m+10$ by 1. Hence $E(A) \leqq 1$ if $A \leqq 54m+22$.

For $m = 5, d = 6, n = -13, c = 4$,

$$U = 120A - 1479, \quad V = 10A - 1, \quad P = 37, \quad W = 15A - 356,$$
$$F = 25A^2 - 10150A + 126685 > 0 \text{ if } A > 393 = 74m + 23.$$

These facts with Lemma 3 of I prove Theorem 1 for $m=5$.

$$\text{For } m = 4, d = 4, n = -10, c = 3,$$

$$U = 4(24A - 231),\ V = 4(2A - 1),\ P = 14,\ W = 12A - 140,$$

$$F = 16[(7A - 37)^2 + (2A - 1)(24A - 231)]$$

$$= 16(A - 16)^2 + 16 \cdot 882 > 0.$$

Finally, let $m=3$. Then $d=6$, $n=-7$, $c=2$,

$$U = 9(8A - 55),\ V = 6A + 1,\ P = 23,\ W = 9A - 128,$$

$$F = 9[(7A - 42)^2 - (6A + 1)(8A - 55)]$$

$$= 9(A^2 - 266A + 1819) > 0,$$

if $A \geqq 259$. For $A < 259$, Theorem 1 was verified by Tables I and II.

4. THEOREM 2. *If* $k=1$, $m \geqq 8$, $E(A) \leqq m-6$ *except for* $A = 54m+16$.

Within every block of Table II the difference of any two consecutive entries is $\leqq 3$. If f is the term free of m in the leader $qm+f$ of any block having $q \neq 16, 46, 52, 54$, then $f+5$ is that of a number t occurring explicitly in the table. Hence $qm+f-1$ is the sum of $m-6$ and $t=(q-1)m+f+5$. For $q=16$, 46, and 52, a like result holds if we replace $f+5$ by $f+6$ and hence replace $m-6$ by $m-7$. Hence the theorem is true for $A < 54m+16$. For $54m+17 \leqq A \leqq 199m+37$, it is true by Lemmas 3 and 4 of I.

For $A \geqq 199m+38$, $E=m-6$, we have $d=8$,

$$U = 24mA - 63m^2 + 12m + 36,\ V = 2mA - m^2 + 8m + 4,$$

$$P = 11m + 2, \qquad W = 3mA - 23m^2 - 4m + 4,$$

$$F = m^2A^2 - 200m^3A - 48m^2A + 562m^4 - 84m^3$$

$$- 264m^2 - 48m.$$

Then $F>0$, in fact for $A = 198m+44+t$, $t \geqq 0$.

5. *Excess E_s when $k=1$.*

THEOREM 3. *When $k=1$, $E_s=0$ if $m\leqq 7$, $s\geqq 5$, and if $m>7$, $s\geqq m-2$, while $E_s=m-s-2$ if $m>7$, $5\leqq s\leqq m-2$.*

If $m\leqq 5$, this follows from Theorem 1. If $m>7$, $E_5\leqq m-7$ by Theorem 2 and the fact that $54m+16$ is the sum of the polygonal number $3m+3$ and the entry $51m+13$ of Table II. Since the polygonal numbers >1 exceed $m-1$, the summands yielding $m-2$ are all 1, whence $E_s(m-2)=m-2-s$. Hence $E_5=m-7$. If $s>5$, regard $s-5$ of the values 1 as polygonal numbers; hence $E_s\leqq m-7-(s-5)$. This proves Theorem 3 except for $m=6$ and 7.

For $m=6$ or 7, we seek a constant C_m such that $E_4(A)\leqq 1$ for every $A\geqq C_m$. When r takes the values 0 and $E=1$, and b takes 3 or 4 consecutive odd values according as $m=6$ or $m=7$, the values of $b+r$ include a complete set of residues modulo m, whence $d=6$ or 8, respectively.

In the discussion in § 3 for $m\geqq 7$, we had $d=6$, $E=m-5$. Hence it is valid here for $m=6$, $d=6$, $E=1$, and shows that $E_4(A)\leqq 1$ if $A\geqq 74m+28$. Likewise, the work at the end of § 4 is valid here for $m=7$, $d=8$, $E=1$, and shows that $E_4(A)\leqq 1$ if $A\geqq 198m+44$.

It remains to treat the values of A below these two limits. From Table II we readily deduce a list of the sums by five of $m-1$ and polygonal numbers and then verify for $m=6$ and $m=7$ that the list includes all positive integers $\leqq 54m+25$. From thence to $74m+28$, $E_4(A)\leqq 1$ if $m=6$ or 7 by Lemma 3 of I. If $m=7$ and $74m+20\leqq A\leqq 199m+37$, $E_4(A)\leqq 1$ by Lemma 4 of I. This completes the proof of Theorem 3.

6. *Excess E when $k=2$.* The values of (1) for $k=2$, $x\geqq 0$ are $m-1$, $3m-2$, and all the polygonal numbers of order $m+2$. Evidently $E(m-2)=m-6$. For $m=7$ the sums by four of the values mentioned were tabulated to 395; the only consecutive integers missing from the list are 393–4; hence $E(394)=2$, $E(A)\leqq 1$ if $A<394$. For $m=6$, $E(60)=1$. Hence E is not smaller than the value shown in

THEOREM 4. *For* $k=2$, $E=m-6$ *if* $m\geqq 8$, $E=2$ *if* $m=7$, $E=1$ *if* $m=6$, $E=0$ *if* $m=3$, 4, *or* 5. *If* $m=7$, $E(A)\leqq 1$ *except for* $A=394$.

Conditions (9) are satisfied if $A\geqq 16m$. For $r=0, 1, \cdots, E$, we require that $b+r$ shall include a complete set of residues modulo m. When $m\geqq 8$, this will be true if b takes the values β, $\beta+2$, $\beta+4$, $\beta+6$; when $m=5$, also $\beta+8$; when $m=6$ or 7, only the first three.

First, let $m\geqq 8$. Then $d=8$,

$$U = 24mA - 135m^2 + 36m + 36, \qquad P = 9m + 2,$$

$$V = 2mA - m^2 + 8m + 4, \quad W = 3mA - 27m^2 + 4,$$

$$F = m^2A^2 - 112m^2A - 40m^2A + 706m^4 + 188m^3 - 152m^2 - 48m.$$

Evidently $F>0$ if $A\geqq 112m+40$. For smaller A's exceeding $54m+16$, Lemmas 3 and 4 of I show that $E(A)\leqq m-6$ if $m\geqq 7$.

LEMMA 2. *If* $m\geqq 7$ *in Table* II, $E(A)\leqq m-6$ *except for* $A=9m+6$, $20m+9$, *and* $54m+16$.

From each block we suppress all entries down to and including the last entry which differs by 3 from the next entry. Proceed as in § 4. We now have the further exceptions $q=9$ and 20. Also

$$9m + 3 = m - 6 + t_1, \quad 9m + 5 = 1 + t_2,$$

$$20m + 8 = m - 6 + t_3, \quad 54m + 15 = m - 6 + t_4,$$

where the t_i occur in the table.

For $m\geqq 8$, Theorem 4 now follows since

$$9m + 6 = 1 + 2(3m + 3) + 3m - 2 + 1, \quad 20m + 9 = m + 2 + 6m + 4 + 10m + 5 + 3m - 2,$$

$$54m + 16 = 2(3m + 3) + 45m + 10 + 3m - 2 + 2.$$

For $m=7$, $E=2$, $d=6$, $P=37$,

$$U = 168A - 6327, \quad V = 14A - 3, \quad W = 21A - 962,$$

$$F/7 = 7A^2 - 826A + 131149,$$

and $F>0$ for every $A>0$. But if we attempt to use $E=1$, we have $d=8$; the discussion for $m\geqq 8$ now applies also for $m=7$ except when $A=54m+16=394$. This proves both parts of Theorem 4 if $m=7$.

Next, let $m=6$. Then $E=1$, $d=6$, $P=32$,

$$U = 9(16A - 512), \quad V = 4(3A + 1), \quad W = 18A - 704,$$
$$F/36 = (7A - 116)^2 - (3A + 1)(16A - 512) = (A - 52)^2 + 11264.$$

The minor conditions in I are all satisfied if $A\geqq 44$. For $m=6$, Table II includes all numbers less than 44 except $33=1+2(3m-2)$.

Let $m=5$. Then $E=0$, $d=10$, $P=67$,

$$U = 120A - 3159, \quad V = 10A + 9, \quad W = 15A - 956,$$
$$F/25 = (A - 703)^2 - 457878 > 0 \text{ if } A \geqq 1380.$$

It was verified that every integer $\leqq 1380$ is a sum by four of 4, 13, and polygonal numbers of order 7, whence $E=0$.

Let $m=4$. Then $n=-18$, $c=10$, $A=2a-9b+40$. Our general method requires that a and b be odd and applies only when A is odd. By (10) with $r=0$, $b\equiv A \pmod 4$. Hence $d=4$, $P=6$,

$$U = 4(24A - 495), \quad V = 4(2A + 1), \quad W = 4(3A - 63),$$
$$F/16 = (7A - 61)^2 - (2A + 1)(24A - 495) = A^2 + 112A + 4216,$$

whence $F>0$ for every $A\geqq 0$. The minor conditions in I are all satisfied if $A\geqq 28$. For $m=4$, Table II includes all integers $\leqq 28$ except $26=m+2+2(3m-2)$.

Next, let A be even. Since b must be even, our previous method is not applicable.

LEMMA 3.* *If $b=2B$, a is even, $a-B^2$ is a sum of three squares, and*

---

* Legendre, *Théorie des Nombres*, ed. 3, II, No. 628, with omission of denominator 2 in No. 629.

$$4B^2 + 2B + 1 > 3a, \tag{12}$$

then $a = \sum\alpha^2$, $b = \sum\alpha$ *have solutions* $\alpha, \beta, \gamma, \delta$ *in integers* $\geqq 0$.

Write $A = 2S$. Then $a = S + 9B - 20$. Inserting this value of a into (12) and multiplying by 16, we see that (12) holds if

$$8B > 25 + u^{1/2}, \quad u = 48S - 351 \geqq 0. \tag{13}$$

Similarly, $B^2 < a$ if

$$2B < 9 + v^{1/2}, \quad v = 4S + 1. \tag{14}$$

Take $B \equiv 2 - S \pmod 4$. Then $a \equiv 2 \pmod 4$, $a - B^2 \equiv 1$ or $2 \pmod 4$, according as B is odd or even. Hence $a - B^2$ is a sum of three squares, and all conditions in Lemma 3 are satisfied.

To insure the choice of B as a prescribed residue modulo 4, we require that the difference between the limits for B, determined by (13) and (14), shall exceed 4. The resulting inequality reduces as usual to

$$S^2 - 340S + 2440 > 0, \quad S \geqq 333.$$

It remains only to verify Theorem 4 for $m = 4$ when A is even and < 666. This was done above when $A \leqq 28$. Table II includes all integers from 28 to 238 except 236; Table I includes all from 235 to 666.

7. *Excess when* $k \geqq 3$. Evidently $E(m-2) = m - 6$. If $m = 7$,

$$394 = 54m + 16 = 6m - 3 + 2(10m + 5) + 28m + 8 + 1.$$

Hence Theorem 4 implies the cases $m \geqq 7$ of

THEOREM 5. *For* $k \geqq 3$, $E = m - 6$ *if* $m \geqq 7$, $E = 1$ *if* $m = 6$.

That $E = 1$ if $m = 6$ follows from Theorem 4 and a result to be proved in a later paper on extended polygonal numbers. It was verified that 116 (or 980) is the least positive integer which is not a sum by four of the values of (1) for $x \geqq 0$, $k = 4$ (or $k = 5$). For $x \geqq 0$, the values > 1 of (1) are all $\leqq m - 1$. Hence the proof in § 5 shows that Theorem 3 holds also when $k > 1$.

8. THEOREM 6. *If* $m=6$, $k=5$, $E(A)=0$ *except for* $A\equiv 4 \pmod 8$.

If A is odd, our general method applies with

$$d = 6,\ U = 144(A - 119),\ V = 12A + 16,\ P = -4.$$

Then (14) and hence (13) of I holds. Also, (9) hold if $A \leqq 344$, which is below the limit 980 for which $E=0$ (§ 7).

Next, let $A\equiv 2 \pmod 4$. Take $b=2B$ and determine B modulo 3 to make a an integer. Then $a\equiv 2 \pmod 4$. Apply Lemma 3. Then (12) holds and $B^2<a$ if

$$4B > 31 + u^{1/2},\quad 3B < 32 + v^{1/2},\quad u = 4A - 403,$$
$$v = 3A + 4.$$

The difference between these limits for B exceeds d if

$$4v^{1/2} - 3u^{1/2} > R = 12d - 35. \tag{15}$$

Here $d=3$ and (15) holds for every A such that $u>0$.

Finally, let $A\equiv 0 \pmod 8$. Take an odd B such that a is an integer, whence $d=6$. Then $a\equiv 4 \pmod 8$ and Lemma 3 applies. The square of (15) holds if

$$(A - 491)^2 > 65712,\quad A \geqq 748.$$

THE UNIVERSITY OF CHICAGO

A RATIONAL NORMAL FORM FOR CERTAIN QUARTICS*

BY RAYMOND GARVER

A quartic equation with rational coefficients, irreducible in the field R of rational numbers, will have for its Galois group (for the field R) one of the following:

I. The symmetric group of order 24.

II. The alternating group of order 12.

III. A group of order 8 such as the one whose substitutions are 1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423).

IV. A cyclic group of order 4 such as 1, (1324), (12)(34), (1423).

V. The 4-group 1, (12)(34), (13)(24), (14)(23).

It is the purpose of this note to give what seems to be a more direct proof of a theorem of Wiman:†

THEOREM. *Any rational quartic, irreducible in R, whose Galois group for R is either* III, IV, *or* V *above, can be transformed by a rational Tschirnhausen transformation into the form* $y^4+py^2+q=0$, *where p and q are rational.*

It will then follow as a corollary that p and q can be made integral; this can be accomplished by a simple additional transformation.

We may consider the given quartic in the reduced form

$$x^4 + a_2x^2 + a_3x + a_4 = 0, \quad (a_1 = s_1 = 0, a_3 \neq 0). \tag{1}$$

Apply the quadratic transformation

$$y = x^2 + k_1x + k_2. \tag{2}$$

Now the transformed equation will lack its second and fourth terms provided $\sum y=0$, $\sum y^3=0$.‡ The first of these equa-

* Presented to the Society, October 29, 1927.

† Arkiv för Matematik, Astronomi, och Fysik, vol. 3 (1906–7), No. 28.

‡ The summations extend over the 4 roots of (1).

tions, by (2), will be satisfied if $k_2=a_2/2$. To obtain $\sum y^3$ we cube both sides of (2), sum over the four roots of (1), and use Newton's relations applied to (1), namely

$$(3)\qquad \begin{cases} s_2 = -2a_2, \quad s_3 = -3a_3, \quad s_4 = 2a_2^2 - 4a_4, \\ s_5 = 5a_2a_3, \quad s_6 = 6a_2a_4 - 2a_2^3 + 3a_3^2. \end{cases}$$

The equation $\sum y^3=0$ then takes the form

$$(4)\qquad a_3k_1^3 - (a_2^2 - 4a_4)k_1^2 - 2a_2a_3k_1 - a_3^2 = 0.$$

Now it is known* that if x_1, x_2, x_3, x_4 denote the roots of (1), then $y_1=x_1x_2+x_3x_4$, $y_2=x_1x_3+x_2x_4$, $y_3=x_1x_4+x_2x_3$ are the roots of the resolvent cubic

$$(5)\qquad y^3 - a_2y^2 - 4a_4y + 4a_2a_4 - a_3^2 = 0.$$

And if the group of (1) is III, IV, or V, equation (5) will have at least one rational root. In this case (4) will also have a rational root, since the roots of (4) can be expressed rationally in terms of those of (5). It is easily verified that they are

$$(6)\qquad \frac{-a_3}{a_2 - y_1}, \quad \frac{-a_3}{a_2 - y_2}, \quad \frac{-a_3}{a_2 - y_3}.$$

These expressions for the roots of (4) were not, however, by any means obvious; the problem was to exhibit them in some form so that it would be possible to show that (4) has a rational root provided the group of (1) is III, IV, or V. For using this rational value of k_1, together with $k_2=a_2/2$, we have exhibited the rational transformation which leads to the result of the theorem.

Bucht† has pointed out, and it is not difficult to show that if the group of (1) is V, the q of the transformed equation will be not only rational, but the square of a rational number.

THE UNIVERSITY OF ROCHESTER

---

* Dickson, *Elementary Theory of Equations*, 1914, p. 39.

† Arkiv för Matematik, Astronomi, och Fysik, vol. 6 (1910–11), No. 30.

ON THE LAW OF EXCLUDED MIDDLE

BY ALONZO CHURCH*

1. *Introduction.* The purpose of this paper is to discuss the possibility of a system of logic in which the law of excluded middle is not assumed, and also to point out what seem to be errors in a recent paper† in which the conclusion is reached that such a system of logic is self-contradictory.

The law of excluded middle is the logical principle in accordance with which every proposition is either true or false. This principle is used, in particular, whenever a proof is made by the method of *reductio ad absurdum*. And it is this principle, also, which enables us to say that the denial of the denial of a proposition is equivalent to the assertion of the proposition.

The simplest alternative to the inclusion of the law of excluded middle among the principles of logic is its bare omission without assertion of any contrary principle. The effect of such an omission is, of course, to reduce the number of theorems which can be proved and also to render of interest certain theorems otherwise vacuous. We cannot derive theorems which contradict theorems obtained with the aid of the law of excluded middle unless we make some assertion of a principle which contradicts the law of excluded middle.

It is not possible, as an alternative to the law of excluded middle, to assert that some proposition is neither true nor false, because by so doing not only the law of excluded middle would be denied but also the law of contradiction. In fact, to assert that a proposition p is not true and is also not false is to assert at once *not-p* and *not-*(*not-p*) and consequently to assert that *not-p* is both true and false.

* National Research Fellow in Mathematics.

† M. Barzin and A. Errera, *Sur la logique de M. Brouwer*, Académie Royale de Belgique, Bulletins de la Classe des Sciences, (5), vol. 13 (1927), pp. 56–71.

It may be, however, that, by introducing the middle ground between true and false as an undefined term, let us say "tiers" (adopting from the French the word used by Barzin and Errera), and making an appropriate set of assumptions about the existence and properties of tiers propositions, we can produce a system of logic which is consistent with itself but which becomes inconsistent if the law of excluded middle be added.

2. *The Position of L. E. J. Brouwer.* L. E. J. Brouwer proposes that the law of excluded middle should not be regarded as an admissible logical principle,* and expresses, as a basis for his proposal, doubts concerning the truth of this law. He says, for example, that the law of excluded middle has been extended to the mathematics of infinite classes by an unjustified analogy with that of finite classes. He says also that to assert the law of excluded middle is equivalent to asserting the doubtful proposition that every proposed theorem can be either proved or disproved if the proper method be found. The latter point depends, of course, on identifying the truth of a proposition with the possibility of proving the proposition. But it seems more in accord with our usual ideas to think of truth as a property of a proposition independent of our ability to prove it. Consequently we prefer to take the truth of a proposition merely as an undefined term subject to certain postulates, among them, if we choose to include it, the law of excluded middle.

In connection with geometry and other branches of mathematics it is commonly recognized that it is meaningless to ask about the absolute truth of a postulate and that the choice between one of two contrary postulates must be made on the basis of simplicity and serviceability. It seems reasonable to recognize the same thing with regard to the postulates of logic, in particular the law of excluded middle, and to say on this basis that it is meaningless to ask about the truth of the law of excluded middle.

* See, for example, *Intuitionistische Mengenlehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 28 (1919), pp. 203–208.

Taking this point of view, we may accept a system of logic in which the law of excluded middle is assumed, a system in which the law of excluded middle is omitted without making a contrary assumption, and a system which contains assumptions not in accord with the law of excluded middle as all three equally admissible, unless one of them can be shown to lead to a contradiction. If we had to choose among these systems of logic, we could choose the one most serviceable for our purpose, and we might conceivably make different choices for different purposes.

3. *Barzin and Errera's Paper.* Barzin and Errera, however, reach the conclusion (loc. cit.) that the system of logic proposed by Brouwer leads to contradiction. This conclusion we believe to be erroneous, for the following reasons.

The method of the argument is the method of *reductio ad absurdum*. It is assumed that if the law of excluded middle is not accepted then it must be explicitly denied by asserting the existence of tiers propositions, and on this basis contradictory results are obtained. This argument is clearly not effective against one who merely omits the law of excluded middle from his system of logic without assuming any contrary principle, because the insistence that one who refuses to accept a proposition must deny it can be justified only by an appeal to the law of excluded middle, the very principle in doubt. The method of *reductio ad absurdum*, in fact, necessarily employs the law of excluded middle and cannot be used against one who does not admit this law.

The argument of Barzin and Errera would not be effective, however, even against one who asserted the existence of tiers propositions, because in the course of the argument it is necessary to assume what the authors name the principle of excluded fourth, that every proposition is either true, false, or tiers, a principle which seems to be a restricted form of the law of excluded middle. This assumption is defended on the basis of the definition of a tiers proposition as one which is neither true nor false, but this definition, as pointed out above, is not consistent with itself and consequently

cannot be used. If the concept *tiers* is introduced at all, it must be as an undefined term.

If we admit the concept *tiers* and the principle of excluded fourth, the argument of Barzin and Errera is then quite correct. They show that "p is tiers" together with "p obeys the principle of excluded fourth" implies a contradiction; hence that p is not tiers; and hence, by an appeal to the principle of excluded fourth, that p is either true or false. But one who asserts the existence of tiers propositions need not assert the principle of excluded fourth; and one who merely omits the law of excluded middle without introducing the concept *tiers* certainly could not assert the principle of excluded fourth.

It would not be permissible to say that the word *tiers* is to cover all possibilities other than true and false, however many such possibilities there may be. This language is, in fact, equivalent to defining tiers to mean neither true nor false, and is open to the same objections. But the following objection could also be raised. Using p' to stand for "p is tiers," it seems likely that $(p')'$, or p'', would constitute a fourth possibility, $(p'')'$ a fifth possibility, and so on. Uniting these possibilities into one under a new name, let us say p^*, we then have to reckon with the likelihood that $(p^*)'$ gives still another possibility which we may designate by the transfinite ordinal ω. And after this we have to reckon with an $(\omega+1)$th possibility, and so on, so that ultimately we reach any given transfinite ordinal whatever. Any attempt to unite all these possibilities into one must be regarded as extremely doubtful, because of its connection with Burali-Forti's paradox† concerning the ordinal number of the sequence of all transfinite ordinals. This paradox, in fact, compels us to regard as illegitimate the consideration of this sequence as a whole.

HARVARD UNIVERSITY

† C. Burali-Forti, *Una questione sui numeri transfiniti*, Rendiconti del Circolo Matematico di Palermo, vol. 11 (1897), pp. 154–164. Whitehead and Russell, *Principia Mathematica*, vol. III, pp. 73–75.

NOTE ON INTERCHANGE OF ORDER OF LIMITS*

BY T. H. HILDEBRANDT

There exist a number of theorems giving necessary and sufficient conditions for the relation

$$\lim_m \lim_n a_{mn} = \lim_n \lim_m a_{mn},$$

$\{a_{mn}\}$ being a double sequence of real numbers. Most of these theorems are not symmetric in m and n, which is only natural, because as a rule wherever the interchange of order of limits is in question, there is information about the iterated limit in one order, and the existence of the limit in reverse order is desired. Nevertheless it may be of interest to deduce a condition which is symmetric in m and n.

By way of notation, we shall assume that the symbol $\lim_m \lim_n a_{mn}$ implies that for every m, $\lim_n a_{mn}$ exists. On the other hand, we shall use the symbol $\lim_m \underline{\lim}_n a_{mn}$ with the implication that

$$\lim_m \overline{\lim}_n a_{mn} = \lim_m \underline{\lim}_n a_{mn}.$$

Then, obviously, if we deduce necessary and sufficient conditions for the relation

$$\lim_m \overline{\lim}_n a_{mn} = \lim_n \overline{\lim}_m a_{mn},$$

those for the equality mentioned at the outset require in addition the assumption of the existence of $\lim_n a_{mn}$ for every m and of $\lim_m a_{mn}$ for every n.

Let us deduce necessary conditions. Let $\overline{\lim}_m a_{mn} = b_n$ and $\underline{\lim}_m a_{mn} = c_n$, and $\lim_n b_n = \lim_n c_n = d$. Then for every $\epsilon > 0$, there exists an n_ϵ such that when $n \geqq n_\epsilon$, we have

$$|b_n - d| \leqq \epsilon \text{ and } |c_n - d| \leqq \epsilon.$$

From the definition of b_n and c_n we have, for every n, and

* Presented to the Society, September 8, 1927.

every $\epsilon>0$, an $m_{\epsilon n}$ (depending on ϵ and n) such that when $m \geqq m_{\epsilon n}$, we have

$$c_n - \epsilon \leqq a_{mn} \leqq b_n + \epsilon.$$

By combining these two statements, we obtain the following. For every $\epsilon>0$, there exists an n_ϵ such that for every $n_1 \geqq n_\epsilon$, there exists an $m_{\epsilon n_1}$ such that for every $m_2 \geqq m_{\epsilon n_1}$, we have

$$d - 2\epsilon \leqq a_{m_2 n_1} \leqq d + 2\epsilon,$$

i. e., $$|a_{m_2 n_1} - d| \leqq 2\epsilon.$$

For future reference we shall call this Statement A. Similarly, from the fact that $\lim_m \overline{\lim}_n a_{mn} = d$, we have: For every $\epsilon>0$ there exists an m_ϵ such that for every $m_1 \geqq m_\epsilon$, there exists an $n_{\epsilon m_1}$ such that for every $n_2 \geqq n_{\epsilon m_1}$, we have

$$|a_{m_1 n_2} - d| \leqq 2\epsilon.$$

By combining these two statements into one, and replacing 4ϵ by ϵ, we get the necessary condition desired, viz.: For every $\epsilon>0$, there exists an n_ϵ and an m_ϵ such that for every $n_1 \geqq n_\epsilon$ and $m_1 \geqq m_\epsilon$, there exists an $m_{\epsilon n_1}$ and an $n_{\epsilon m_1}$ such that for every $m_2 \geqq m_{\epsilon n_1}$ and $n_2 \geqq n_{\epsilon m_1}$, we have

$$|a_{m_1 n_2} - a_{m_2 n_1}| \leqq \epsilon.$$

This condition is also sufficient. For suppose the condition satisfied. Then for a particular m_1 and n_2 chosen in accordance with the specifications, and for every $n_1 \geqq n_\epsilon$ and every $m_2 \geqq m_{\epsilon n_1}$, we have

$$a_{m_1 n_2} - \epsilon \leqq a_{m_2 n_1} \leqq a_{m_1 n_2} + \epsilon.$$

From this we conclude that for every $n_1 \geqq n_\epsilon$, the greatest and the least of the limits b_n and c_n satisfy the conditions

$$a_{m_1 n_2} - \epsilon \leqq c_{n_1} \leqq b_{n_1} \leqq a_{m_1 n_2} + \epsilon.$$

This has as consequence that if n_1 and n_0 are greater than or equal to n_ϵ, then

$$|b_{n_0} - b_{n_1}| \leqq 2\epsilon, \quad |c_{n_0} - c_{n_1}| \leqq 2\epsilon, \text{ and } |c_{n_1} - b_{n_1}| \leqq 2\epsilon.$$

Hence $\lim_n c_n$ and $\lim_n b_n$ exist and are equal, i.e. $\lim_n \overline{\lim}_m a_{mn}$ exists. From the symmetry of the condition, we conclude that $\lim_m \overline{\lim}_n a_{mn}$ exists also. The identity of the two limits is then a consequence of the condition of our theorem and Statement A.

We note finally that the Cauchy condition for convergence of the double limit, $\lim_{mn} a_{mn}$, is the special case of our condition in which $m_{\epsilon n_1}$ and $n_{\epsilon m_1}$ are independent of n_1 and m_1 respectively, and can therefore be taken as m_ϵ and n_ϵ, respectively.

UNIVERSITY OF MICHIGAN

ON BOUNDED REGULAR FRONTIERS IN THE PLANE*

BY W. A. WILSON

1. *Introduction.* The term *regular frontier* has been introduced by P. Urysohn† to designate a continuum which is the frontier of two or more components of its complement. Regular frontiers in the plane have been discussed by various authors. A. Rosenthal‡ has shown that a continuum which is the union of two bounded continua that are irreducible between the same pair of points and have no other common points is a regular frontier. R. L. Moore§ has given necessary and sufficient conditions that a bounded continuum be a regular frontier whose complement has exactly two components. C. Kuratowski‖ has given necessary conditions for a continuum to be a regular frontier which is the frontier of *every* component of its complement.

---

* Presented to the Society, October 29, 1927.

† P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7, p. 98.

‡ A. Rosenthal, *Teilung der Ebene durch Irreduzible Kontinua*, Sitzungsberichte der Münchener Akademie, 1919.

§ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6, pp. 203–213.

‖ C. Kuratowski, *Sur les coupures du plan*, Fundamenta Mathematicae, vol. 6, pp. 130–145.

In the present article Rosenthal's theorem is generalized to cover the case of two continua having a finite or enumerably infinite set of closed sets in common and irreducible between each pair. With this result and a theorem by the author† given elsewhere, it is possible to formulate necessary and sufficient conditions for a bounded continuum to be a regular frontier. The principal theorems are to be found in §§5–7.

2. *Notation.* Besides the ordinary notation of the aggregate theory the following special notation and terminology will be used.

The whole plane will be denoted by Z. If A is a sub-set of C, the set of inner points of A relative to C will be denoted by A.* If A and B are two sets in the plane Z without common points and C separates A from B (i.e., $C \cdot A = C \cdot B = 0$ and every continuum in the plane which contains points of both A and B also contains one or more points of C), we say that C is an $S(A, B)$. If F is a frontier set, a component of $Z-F$ which has the frontier F will be called a *principal* component.

The statement "C is a continuum irreducible between the sets A and B" will apply not only to the case that $A+B \subset C$, but also to the case that C is irreducible between each point of $C \cdot A$ and each point of $C \cdot B$.

3. *Some Auxiliary Theorems.* Let $F=H+K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$, such that both H and K are irreducible between each pair. The following properties of F are either well known or so easily established that the proofs are omitted.

(a) *If the number of sets* $\{\alpha\}$ *is greater than two, both* H *and* K *are indecomposable.*

(b) *If* C *is a sub-continuum of* H *or of* K, $F-C$ *is connected.*

(c) $H^* = F-K$ *and* $K^* = F-H$ *are connected; also* $\overline{H}^* = H$ *and* $\overline{K}^* = K$.

† W. A. Wilson, *On irreducible cuts of the plane between two points.* (To appear soon in Annals of Mathematics.)

If F lies in a plane Z, certain properties expressing relations between F and the rest of the plane can be proved without difficulty.

(d) *If $H \cdot K$ has at least n components, there are at least n components of $Z-F$ which have frontier points on both H^* and K^*. If $H \cdot K$ has an infinity of components, the components of $Z-F$ having frontier points on both H^* and K^* are enumerably infinite.*

These statements are proved by adding to H and K those components of $Z-F$ which do not have frontier points on both H^* and K^* and applying a theorem of S. Straszewicz.†

(e) *There is a bounded continuum P which is an irreducible $S(H^*, K^*)$ and contains $H \cdot K$.*

To establish this we observe that, since H^* and K^* are connected, there is a bounded continum Q which is an $S(H^*, K^*)$ by a theorem of Knaster and Kuratowski.‡ But by a theorem of Mazurkiewicz,§ Q contains a subcontinuum P which is an irreducible $S(H^*, K^*)$. Moreover $P \supset H \cdot K$, since every point of $H \cdot K$ is a common limiting point of H^* and K^*.

It may be added that every bounded irreducible $S(H^*, K^*)$ is a continuum containing $H \cdot K$.

4. LEMMA. *Let $F=H+K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$ between each pair of which both H and K are irreducible. Let R be a component of $Z-F$ such that $\overline{R}$ contains a bounded continuum containing points of two or more sets α but no other points of F. Then F is the frontier of R.*

PROOF. The principles of inversion permit us to demonstrate the lemma on the assumption that R is bounded. Let

---

† S. Straszewicz, *Über die Zerschneidung der Ebene*, Fundamenta Mathematicae, vol. 7, p. 174.

‡ B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2, p. 236.

§ S. Mazurkiewicz, *Sur un ensemble G*, etc., Fundamenta Mathematicae, vol. 1, p. 63.

C be the continuum referred to above. Then R contains a closed polygon J (whose interior is I) so large that $C-C\cdot I$ has no component containing points of more than one α. This is easily seen since the number of sets α is finite. On the other hand, if α_1 and α_2 are two sets met by C, there are sub-continua A_1 and A_2 of $C-C\cdot I$ irreducible between α_1 and α_2, respectively, and J, and $A_1\cdot A_2=0$. Let M be an arc of J irreducible beween A_1 and A_2. Then $Q=A_1+M+A_2$ is irreducible between α_1 and α_2 and does not meet $F-(\alpha_1+\alpha_2)$.

Since Q and H are both irreducible between α_1 and α_2 and have only points of these sets in common, while Q is not the union of two indecomposable continua, it follows by an extension of Rosenthal's theorem† that $Z-(Q+H)$ has two principal components S_1 and S_2. Similarly, let T_1 and T_2 be the principal components of $Z-(Q+K)$.

As K^* is connected, it lies in but one component of $Z-(Q+H)$; suppose that $K^*\cdot S_1=0$. Since $(F+Q)\cdot S_1=0$ and S_1 has frontier points on M, $R\cdot S_1\neq 0$ and $S_1\subset R$. Likewise either T_1 or T_2, say T_1, is a part of R. As $F=H+K$, every point of F is a frontier point of either S_1 or T_1, and a fortiori of R.

5. THEOREM. *Let $F=H+K$ be the union of two bounded continua and let $H\cdot K$ be the sum of a finite number n of closed sets $\{\alpha\}$ between each pair of which both H and K are irreducible. Then the number of principal components of $Z-F$ is at least n.*

PROOF. Let R_i be any component of $Z-F$ having frontier points on both H^* and K^*, let P be a bounded irreducible $S(H^*, K^*)$, and let $P_i=P\cdot R_i$. Suppose that $\overline{P}_i$ has a sub-continuum containing points of more than one set α for only k values of i, say $i=1, 2, \cdots, k$, where $k<n$. We first show that $k\geqq 1$. For, if α' denotes any α and α'' the sum of the remaining sets α, the continuum P contains a connected

† W. A. Wilson, *On the separation of the plane by irreducible continua*, this Bulletin, vol. 33, pp. 733–744, §5.

set having no points of α' or α'', but limiting points on both. This connected set must be a part of some P_i.

In some work done elsewhere† it was shown that there is a continuum Q, which is a bounded irreducible $S(H^*, K^*)$, constructed as follows: Q is the union of k arcs $\{A_i\}$, each A_i lying in R_i, $i=1, 2, \cdots, k$, and k continua $\{B_i\}$, such that $B_i \cdot B_{i'}=0$ if $i \neq i'$, each B_i is irreducible between an end of A_i and one of A_{i+1} ($i+1=1$, if $i=k$) and meets no other points of $\sum_1^k A_i$, and $H \cdot K \subset \sum_1^k B_i \subset P$. Moreover, in the demonstration referred to, the circle used may be replaced by a polygon so large that, if I_i is its interior, no component of $\overline{P}_i - \overline{P}_i \cdot I_i$ has points on two sets α. Consequently any connected sub-set of any B_i having limiting points on two sets α, but no points on F, must lie in some R_i, $i>k$.

Since $n>k$, some B_i contains points of two or more sets α. Hence for some $i>k$, there is a connected sub-set j of B_i having limiting points on two sets α, but containing no points of F.‡ Then j lies in some P_i, where $i>k$ by the last part of the previous paragraph. Therefore $\overline{P}_i$ contains points of more than one set α, contrary to the assumption at the beginning of the proof.

Therefore, for at least n values of i, $\overline{P}_i$, and consequently $\overline{R}_i$, contains a continuum having points on more than one set α. Then §4 shows that F is the frontier of each such R_i. Hence the theorem is proved.

COROLLARY. *Let $F=H+K$ be the union of two bounded continua and for every integer n let $H \cdot K$ be the sum of n closed sets such that both H and K are irreducible between each pair. Then the number of principal components of $Z-F$ is infinite.*

† See reference to paper by the author under §1, §§6 and 7. The hypothesis in these sections requires R to be a principal component of $Z-F$, but the demonstration only requires that R have frontier points on both H^* and K^*.

‡ Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24, Theorem 1.

6. THEOREM. *For the bounded decomposable continuum F to be the frontier of exactly n components of its complement, it is necessary and sufficient that F be the union of two continua H and K such that $H \cdot K$ is the sum of n, but of no finite number greater than n, closed sets between each pair of which both H and K are irreducible.*

PROOF. If F is decomposable and $Z-F$ has n principal components, it has been shown elsewhere† that F is the union of two continua H and K such that $H \cdot K$ is the sum of a finite number, greater than or equal to n, of closed sets between each pair of which both H and K are irreducible. On the other hand, if $H \cdot K$ can be expressed as the sum of n closed sets between each pair of which both H and K are irreducible, there are at least n principal components of $Z-F$ by §5.

The combination of these statements gives the theorem.

REMARKS. The decomposition of $H \cdot K$ into n closed sets given above is unique. For, if there were two different decompositions into n closed sets with the properties assigned, say $H \cdot K = \sum_1^n \alpha_i$ and $H \cdot K = \sum_1^n \beta_i$, some β_i, say β_1, would contain points of more than one set α_i. Let $\beta_1 \cdot \alpha_1 \neq 0$ and let $\beta_{11} = \beta_1 \cdot \alpha_1$ and $\beta_{12} = \beta_1 \cdot (H \cdot K - \alpha_1)$. Then $H \cdot K = \beta_{11} + \beta_{12} + \sum_2^n \beta_i$ is a decomposition into $n+1$ closed sets between each pair of which both H and K are irreducible. We then have the contradiction that $Z-F$ has at least $n+1$ principal components.

7. THEOREM. *For the bounded decomposable continuum F to be the frontier of an infinity of components of its complement, it is necessary and sufficient that F be the union of two continua H and K such that for every integer n the set $H \cdot K$ is the sum of n closed sets between each pair of which H and K are irreducible.*

PROOF. The necessity of these conditions was shown in

† See reference to paper by the author under §1.

the proof of the theorem referred to in §6. That they are sufficient follows from §5, Corollary.

8. *Two Examples.* It might be inferred that the statements of the theorems in §§6 and 7 are unnecessarily complicated and that, if $H \cdot K$ is the sum of an infinite set of closed sets between each pair of which both H and K are irreducible, then $Z-F$ has an infinite set of principal components. This is not in general true. The following examples show the existence of two continua H and K having these properties, but such that $Z-F$ has no principal component in Ex. I and but two principal components in Ex. II.

EXAMPLE I. Let $Q=OABC$ be a closed rectangle, such that the lengths of OA and AB are 1 and 1/2 unit, respectively. Let q be the frontier of Q. Let M be a Cantor set extending from O to A, whose complementary open intervals $\{I_n\}$ are ordered according to size.

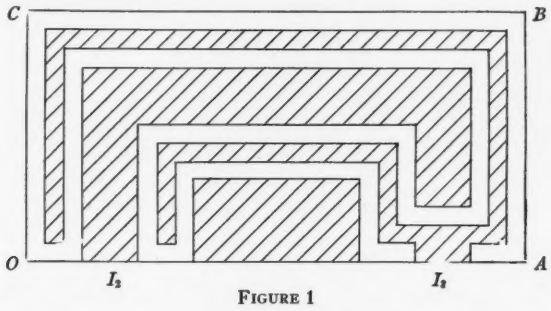

FIGURE 1

Now let a finite ordered set of closed rectangles, each of width d and of length greater than d, such that each rectangle has in common with the one preceding and the one following two different squares of side d but no point in common with any other rectangle, be called a *band*† of width d. It is

† For a complete discussion of the "method of bands" as applied to indecomposable continua see B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3, pp. 247–287. See also a memoir by K. Yoneyama, Tôhoku Mathematical Journal, vol. 12, pp. 60–62.

easily seen that there is in Q a unique band B_1 of width $1/3$, which is contiguous to all points of $q-I_1$ and whose frontier q_1 is the union of $q-I_1$ and a broken line b_1 meeting q only in the end-points of I_1. Then $Q-B_1$ consists of I_1 and a simply connected region E_1 whose frontier is I_1+b_1; let $G_1=Q-B_1$.

Likewise, in B_1 there is a unique band B_2 of width $1/3^2$ whose frontier q_2 is the union of q_1-I_1 and a broken line b_2 meeting q_1 only in the end-points of I_2. Then $G_2=B_1-B_2$ consists of I_2 and a region E_2 whose frontier is I_2+b_2. This construction can be repeated indefinitely; in the figure the shaded area is $G_1+G_2+G_3$; the unshaded with its border is B_3.

Set $H=Q-\sum_1^\infty G_n=\prod_1^\infty B_n$. Obviously H is a continuum. The Cantor set M may be regarded as the sum of an infinity of closed sets without common points. These are the set $O+A$, an enumerable set of sets each consisting of the end-points of an interval I_n, and a non-enumerable set of sets each of which is one of the other points of M. Let α be any of these sets. It is comparatively easy to show that H is irreducible between each pair of sets α.

Let K be the continuum symmetrical to H with respect to OA. Then K is also irreducible between each pair of sets α and $H\cdot K=\sum\alpha$. But each component of $Z-F$ has for its frontier a pair of continua of condensation of H and K.

EXAMPLE II. This is a variation of Ex. I. Take a closed rectangle Q whose length and width are $43/27$ units and 1 unit respectively. Let ab be one side and let the points c, d, e and f divide ab into five intervals of lengths $ac=1/3$; $cd=2/27$; $de=1$; $ef=2/27$; $fb=1/9$. Let $\{I_k\}$ be an enumerable set of open intervals divided into three classes as follows: $\{I_{3n}\}$ is the set of complementary intervals of a Cantor set M in the interval de ordered as in Ex. I; $\{I_{3n-2}\}$ and $\{I_{3n-1}\}$ are two other sets, of which the first are $I_1=cd$ and $I_2=ef$, and the others will be defined later. Let q be the frontier of Q.

For $k=1, 2, 3$, let B_k, q_k, b_k, E_k, and G_k be defined as in Ex. I. For $n=2$, let $I_k=I_{3n-2}=I_4$ be an open interval of length $1/9$ at the center of one of the longest segments of

b_1 perpendicular to, but not meeting, ab. Then B_4 is a band of width $1/3^4$ contiguous to q_3-I_4. This gives q_4, b_4, E_4, and G_4. In this case, we note that E_1+G_4 is a simply connected region whose frontier is $I_1+(b_1-I_4)+b_4$. In like manner, we let I_5 be an open interval of length $1/9$ at the center of one of the longest segments of b_2 perpendicular to and not meeting ab and define B_5, q_5, b_5, E_5, and G_5. Here E_2+G_5 is a simply connected region whose frontier is $I_2+(b_2-I_5)+b_5$.

We now return to I_6, which has been defined above. Let this process be repeated indefinitely. Each I_{3n} lies on de; each I_{3n-2} is an open interval of length $1/9$ at the center of one of the longest segments of b_{3n-5} perpendicular to and not meeting ab; and similarly for I_{3n-1}. In the figure the shaded

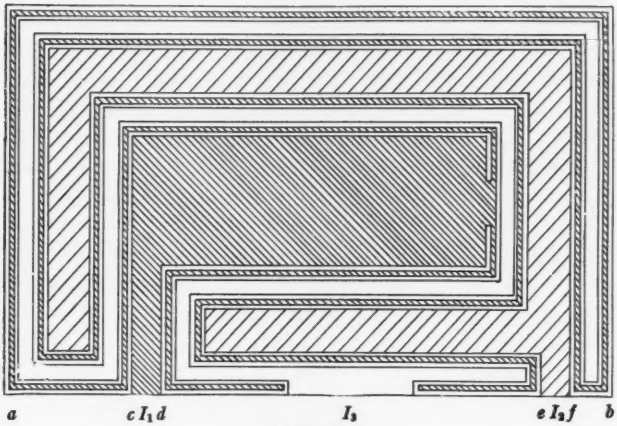

FIGURE 2

areas are E_1+G_4 and E_2; the narrower white area is B_4; and the other white area is E_3. For every value of k, Q is the union of B_k and a finite number of the sets G_k. No two of the sets G_{3n} or of the corresponding regions E_{3n} have common points. For each n, $E_1+G_4+\cdots+G_{3n-2}$ and $E_2+G_5+\cdots+G_{3n-1}$ are simply connected regions whose frontiers

are the union of I_1 and I_2, respectively, with a broken line interior to Q.

Let $H=Q-\sum_1^\infty G_k=\prod_1^\infty B_k$. Obviously H is a continuum. The set $ac+fb+M$ is a closed set which may be regarded as the sum of an infinity of closed sets without common points. These are the set $ac+fb$, an enumerable set of sets each consisting of the end-points of an interval I_{3n}, and a non-enumerable set of sets each of which is one of the other points of M. Let α denote any of these sets. It is comparatively easy to show that H is irreducible between each pair of sets α.

Reflect the above figure about ab and denote corresponding sets by primes. Then $K=H'$ is a continuum irreducible between each pair of sets α and $H\cdot K=\sum\alpha$. The components of $Z-F$, where $F=H+K$, are the exterior of the rectangle $Q+Q'$, the enumerable set of regions $G_{3n}+G'_{3n}$, each of which has as its frontier a pair of continua of condensation of F, and two other regions. These are $\sum_1^\infty G_{3n-2}+\sum_1^\infty G'_{3n-2}$ and $\sum_1^\infty G_{3n-1}+\sum_1^\infty G'_{3n-1}$. Each of these has the frontier F and is therefore a principal component of $Z-F$.

The essential difference between the two examples is that in Ex. I, $H\cdot K$ cannot be divided into any finite number of closed sets between each pair of which H and K are irreducible, while in Ex. II, $H\cdot K$ can be divided into precisely two, but no more, such sets, namely the sets $ac+fb$ and M. Thus Ex. II is in strict accordance with §§6 and 7.

YALE UNIVERSITY

ON THE BEHAVIOR OF INTEGRAL FUNCTIONS IN DISTANT PORTIONS OF THE PLANE*

BY W. B. FORD

1. *Introduction.* In a series of remarkable papers extending over the years 1904–08 Mr. E. W. Barnes† has determined the asymptotic behavior in the neighborhood of the point at infinity of various types of analytic functions. The types considered are distinguished from each other by the forms of their Maclaurin developments, these being regarded as given and forming the starting point of the investigation in each case.‡ In point of method, Barnes makes extensive use of contour integrations and the calculus of residues. In point of results, his emphasis falls rather upon the detailed study of each individual function-type considered than upon the attainment of one or more general theorems which may be regarded as central in character to the subject as a whole. In my book on divergent series (see footnote below) I have already indicated on page 60 a general theorem through which the asymptotic character of a wide variety of special functions, including some of those considered by Barnes, may be readily determined§ and I desire in the present paper to indicate another such theorem, this being of a nature supplementary to the former one. It may be applied

* Presented to the Society, September 8, 1927.

† Now the Bishop of Birmingham (England).

‡ A list of the papers referred to may be found in my book entitled *Studies on Divergent Series and Summability* (Michigan Science Series, vol. II, 1916) page 184. The various function-types considered by Barnes are summarized in his paper entitled *The asymptotic expansion of integral functions defined by Taylor's series* appearing in the Philosophical Transactions of the Royal Society of London, vol. 206 (1906), pp. 249–297.

§ I take this occasion to note that the relation $w=0, 1, 2, 3, \cdots$, occurring in line 6 of the theorem should be corrected to read $w=0, \pm 1, \pm 2, \pm 3, \cdots$.

in particular to the special function designated by Barnes as $G_\beta(z, \theta)$ and defined as follows:

$$(1) \qquad G_\beta(z,\theta) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)(n+\theta)^\beta},$$

where β and θ are any constants, real or complex, except that $\theta \neq 0, -1, -2, \cdots$. In §4 I shall show, in fact, how the theorem may be applied to this special function.

As to method, I find that the elementary theory of differential equations suffices to establish one of the two parts of the theorem. For the other part, I employ the calculus of residues in the manner common to such investigations.

Wherever the symbol $R(z)$ occurs it is to be understood as meaning *the real part of z.*

2. THEOREM.

Part I. $R(z) > 0$. *If, when considered for large (positive integral) values of n, the function g(n) occurring in the coefficient of the power series*

$$(2) \qquad \sum_{n=0}^{\infty} \frac{g(n)}{\Gamma(n+k)} z^n \; ; \; k = \textit{any given constant (real or complex)}$$

may be developed in either a convergent or asymptotic series of the form

$$(3) \qquad g(n) = c_0 + \frac{c_1}{n+k} + \frac{c_2}{(n+k)(n+k+1)} + \cdots + \frac{c_p + \delta(p,n)}{(n+k)(n+k+1)\cdots(n+k+p-1)}$$

where $c_0, c_1, c_2, \cdots$ *are constants and* $\lim_{n\to\infty} \delta(p, n) = 0$, *then the function f(z) defined by the series* (2), *when considered for values of z for which* $R(z) > 0$, *may be developed asymptotically in the form*

$$(4) \qquad f(z) \sim e^z z^{1-k} \left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \right].$$

Part II. $R(z)<0$. *If the function* $g(n)$ *may be regarded as a function* $g(w)$ *of the complex variable* $w=x+iy$ *and as such satisfies the following two conditions*: (a) *it is single valued and analytic throughout the finite w-plane and* (b) *it remains less than a constant when considered for all values of w of large modulus and of real part greater than any arbitrary fixed value* (*positive or negative*), *then the function* $f(z)$ *defined by* (2), *when considered for values of z for which* $R(z)<0$, *may be developed asymptotically in the form*

$$f(z) \sim -\frac{g(-1)}{\Gamma(k-1)}\frac{1}{z}-\frac{g(-2)}{\Gamma(k-2)}\frac{1}{z^2} -\frac{g(-3)}{\Gamma(k-3)}\frac{1}{z^3}-\cdots. \tag{5}$$

It is to be observed at once that under the conditions imposed in either part I or part II of the theorem, the series (2) necessarily has an infinite radius of convergence, thus rendering $f(z)$ a so-called integral function. In this particular the theorem is in contrast to the former one referred to in §1.

In order to prove the theorem, let us begin by considering the special function defined by (2) in case $g(n)=1$; that is, the function which we shall call $F(z)$, defined as follows:

$$F(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+k)}. \tag{6}$$

Upon multiplying both sides of this relation by z^{k-1} and subsequently taking the derivative of each member of the result, making use of the relation

$$\Gamma(n+k)=(n+k-1)\Gamma(n+k-1),$$

we find that

$$\frac{d}{dz}[z^{k-1}F(z)]=\frac{z^{k-2}}{\Gamma(k-1)}+z^{k-1}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+k)} =\frac{z^{k-2}}{\Gamma(k-1)}+z^{k-1}F(z).$$

Thus it appears directly that $F(z)$ satisfies the differential equation

$$F'(z) + \frac{k-1-z}{z}F(z) = \frac{1}{z\Gamma(k-1)}.$$

This equation, being linear and of the first order, is readily solved by elementary methods giving as its general solution

$$F(z) = \frac{e^z z^{1-k}}{\Gamma(k-1)} \int_c^z e^{-z} z^{k-2} dz, \tag{7}$$

where c is an arbitrary constant. In order that this $F(z)$ shall coincide with the particular $F(z)$ defined by (6) we need only to determine c in such a manner that the limit approached by (7) as z approaches zero shall be the first term of (2); that is, shall be $1/\Gamma(k)$. Moreover, this will be the case if we take $c=0$, it being assumed for the moment (in order that the integral involved may assuredly have a meaning) that $R(k)>1$. In fact, with $c=0$ in (7) we have as desired

$$\lim_{z\to 0} F(z) = \frac{1}{\Gamma(k-1)} \lim_{z\to 0} \frac{\int_0^z e^{-z} z^{k-2} dz}{z^{k-1}} = \frac{0}{0}$$

$$= \left[\frac{e^{-z} z^{k-2}}{\Gamma(k-1)(k-1)z^{k-2}}\right]_{z=0} = \frac{1}{\Gamma(k)}.$$

Whenever $R(k)>1$ we may therefore express the given $F(z)$ of (6) as follows:

$$F(z) = \frac{e^z z^{1-k}}{\Gamma(k-1)} \int_0^z e^{-z} z^{k-2} dz. \tag{8}$$

If we now confine ourselves to values of z for which $R(z)>0$, we may write (8) in the form

$$F(z) = \frac{e^z z^{1-k}}{\Gamma(k-1)}\left[\int_0^\infty e^{-z} z^{k-2} dz - \int_z^\infty e^{-z} z^{k-2} dz\right],$$

where it is understood that the integration in the first integral takes place along the positive real axis from $z=0$ to $z=+\infty$,

while in the second integral it takes place in the direction of the positive real axis from $z=z$ to $z=\infty$.

We have now but to recall that

$$\int_0^\infty e^{-z}z^{k-2}dz = \Gamma(k-1)$$

in order to have

$$(9)\quad F(z) = e^z z^{1-k}\left[1 - \frac{1}{\Gamma(k-1)}\int_z^\infty e^{-z}z^{k-2}dz\right];\ R(z) > 0.$$

The expression here appearing is seen to have a meaning not only under the tentative assumption $R(k)>1$, which was introduced above, but for all values (real or complex) of k. In fact, it is analytic throughout the finite k-complex plane, and, inasmuch as the same is true of the expression occurring on the right in (6), it follows that, whatever the value of k, $F(z)$ is given by (9) so long as $R(z)>0$. Moreover, with $R(z)$ thus limited, we see that as mod z increases indefinitely the integral appearing in (9) becomes an infinitesimal of order higher than (mod z)$^{-m}$, where m is any assigned positive integer. Hence, if $R(z)>0$, we may write

$$(10)\qquad F(z) = e^z z^{1-k}[1+\eta(z)];\qquad \lim_{|z|\to\infty} z^m\eta(z) = 0,$$

where $m=0, 1, 2, 3, \cdots$. To state this in the language of the theory of asymptotic series, we thus may write when $R(z)>0$,

$$F(z) \sim e^z z^{1-k}\left[1 + \frac{0}{z} + \frac{0}{z^2} + \frac{0}{z^3} + \cdots\right],$$

or simply

$$(11)\qquad F(z) \sim e^z z^{1-k}.$$

This preliminary result having been established, we proceed to establish part I of the theorem. In order to avoid unnecessary repetition of statement, let it be henceforth understood until the contrary is stated that the only values of z under consideration are those for which $R(z)>0$.

Having chosen an arbitrarily large positive integer m, we may write the function $f(z)$ defined by (2) in the form

$$f(z) = \phi(z,m) + \psi(z,m), \tag{12}$$

where

$$\phi(z,m) = \sum_{n=0}^{m-1} \frac{g(n)}{\Gamma(n+k)} z^n, \quad \psi(z,m) = z^m \sum_{n=0}^{\infty} \frac{g(m+n)}{\Gamma(n+k+m)} z^n .$$

By choosing m sufficiently large, the coefficient $g(m+n)$ may be developed for all values of n in the form (3). If this be done, $\psi(z, m)$ takes the following form:

$$\psi(z,m) = c_0 z^m \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+k+m)} + z^m \sum_{r=1}^{p} c_r \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+k+m+r)} + R(z,m,p), \tag{13}$$

where

$$R(z,m,p) = z^m \sum_{n=0}^{\infty} \frac{\delta(m+n,p)}{\Gamma(n+k+m+p)} . \tag{14}$$

Considering now the first two expressions appearing on the right in (13), if we apply (11) to the first of these, using the appropriate value of k, namely, $k=k+m$, we see that for values of z of large modulus it becomes asymptotic to $c_0 z^m e^z z^{1-k-m} = c_0 e^z z^{1-k}$. Similarly, the second of the expressions in question consists of p series of which the rth is seen to be asymptotic to $c_r e^z z^{1-k-r}$, when mod z is large. Hence, $\psi(z, m)$ is of the form

$$\psi(z,m) = \Omega(z,m,p) + R(z,m,p) \tag{15}$$

where

$$\Omega(z,m,p) \sim e^z z^{1-k} \left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_p}{z^p} \right]. \tag{16}$$

We shall now show that when mod z is large $\psi(z, m)$ may be developed asymptotically in the analogous series (instead of polynomial) form as follows:

$$(17) \qquad \psi(z,m) \sim e^z z^{1-k}\left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right].$$

In order to do this, it suffices in view of (15) and (16) to show that, having chosen an arbitrarily small positive quantity ϵ, we may determine m such that, by subsequently taking $\rho =$ mod z sufficiently large, we may write

$$(18) \qquad |R(z,m,p)| < \frac{K\epsilon e^\rho \rho^{1-k}}{\rho^p} = K\epsilon e^\rho \rho^{1-p-k}\,;\ K = \text{const.}$$

This relation, in fact, may be brought about as follows: For the given ϵ choose m so large that $|\delta(m, p)| < \epsilon$. Then we shall likewise have necessarily $|\delta(m+n, p)| < \epsilon$; $n = 0, 1, 2, 3, \cdots$. From the definition (14) of $R(z, m, p)$, we may then write

$$|R(z,m,p)| < \rho^m \epsilon \sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(n+k+m+p)}.$$

Applying (10) with $k+m+p$ used for k, we thus have

$$\begin{aligned} |R(z,m,p)| &< \rho^m \epsilon e^\rho \rho^{1-(k+m+p)}[1 + \eta(m,p,z)] \\ &= \epsilon e^\rho \rho^{1-p-k}[1 + \eta(m,p,z)], \end{aligned}$$

where $\lim_{|z|\to\infty} \eta(m, p, z) = 0$. This result at once yields the desired relation (18), thus establishing (17).

We have now but to observe that $\phi(z, m)$, being a polynomial in z, is such that

$$(19) \qquad \phi(z,m) \sim e^z z^{1-k}\left[0 + \frac{0}{z} + \frac{0}{z^2} + \frac{0}{z^3} + \cdots\right],$$

in order to obtain from (12), (17) and (19) the desired result (4) of part I of the theorem.

We therefore proceed to the proof of part II of the theorem. For this we shall employ the following fundamental statement from the calculus of residues:*

* The statement may be regarded as an immediate corollary of the fundamental theorems of Cauchy in the theory of functions of a complex variable. Details of a proof may be found, however, on page 9 of my *Divergent Series*, etc. already mentioned.

"If $P(w)$ and $Q(w)$ are any two functions of the complex variable w both of which are single valued and analytic throughout a region A of the w-plane and of which $Q(w)$ vanishes within A only at the points $w=\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ which are zeros of the first order, and if C_n denotes any closed contour lying within A and including the points $w=\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, then we may write

$$\frac{1}{2\pi i}\int_{C_n}\frac{P(w)}{Q(w)}dw = \sum_{n=1}^{n}\frac{P(\lambda_n)}{Q'(\lambda_n)}, \tag{20}$$

where the indicated integration is taken about the entire contour C_n in the positive direction."

Formula (20) being premised, let us take for our present purposes

$$P(w) = \frac{\pi g(w)(-z)^w}{\Gamma(w+k)}, \qquad Q(w) = \sin \pi w,$$

where $g(w)$ is the function described in part II of the theorem, z being regarded for the present as having any fixed value. Moreover, for the contour C_n, let us take the rectangle in the w-plane formed by the straight lines

$$w = a - \tfrac{1}{2} + iy, \quad w = \tfrac{1}{2} + 2n + iy, \quad w = x \pm ij,$$

where a is any arbitrarily large negative integer, n is an arbitrarily large positive integer and j is any arbitrarily large positive quantity. Applying (20), we then obtain the relation

$$\begin{aligned}\frac{1}{2i}\int_{C_n}\frac{g(w)(-z)^w dw}{\Gamma(w+k)\sin \pi w} &= \sum_{n=a}^{2n}\frac{g(n)z^n}{\Gamma(n+k)} \\ &= \sum_{n=a}^{-1}\frac{g(n)z^n}{\Gamma(n+k)} + \sum_{n=0}^{2n}\frac{g(n)z^n}{\Gamma(n+k)},\end{aligned} \tag{21}$$

the last summation on the right thus becoming the sum of the first $2n$ terms of the given series (2). As to the integral on the left, we shall assume that the function $(-z)^w$ appearing in the integrand is rendered precise in meaning for any given

value of $z=\rho(\cos\phi+i\sin\phi)$ through the following convention:

$$(22)\qquad (-z)^w = e^{w\log(-z)} = e^{w[\log\rho+i(\phi+\pi)]}\ ;\ -2\pi<\phi\leqq 0.$$

Supposing at first that z is real and negative; that is, that $\phi=-\pi$, we now proceed to study in detail the integral appearing in (21).

First, along the side of the rectangle upon which $w=x+ij$, we have $dw=dx$ and $\sin\pi w=\sinh\pi j(\sin\pi x\ \mathrm{ctnh}\,\pi j+i\cos\pi x)$, so that, if we designate by I the contribution to the integral from this side, we may write

$$(23)\qquad I=\frac{(-z)^{ij}}{2i\sinh\pi j}\int_{2n+1/2}^{a-1/2}\frac{g(x+ij)(-z)^x dx}{\Gamma(x+k+ij)(\sin\pi x\ \mathrm{ctnh}\,\pi j+i\cos\pi x)}.$$

Now, as j is allowed to increase indefinitely, $\sinh\pi j$ becomes infinite like $e^{\pi j}$, while $|\Gamma(x+k+ij)|$ vanishes to as high an order as that of $e^{-\pi j/2}$;* also we have $\lim_{j\to\infty}\mathrm{ctnh}\,\pi j=1$. Hence, recalling the condition (b) of part II of the theorem, we see that $\lim_{j\to\infty}I=0$.

Similarly, the contribution to the contour integral in (21) arising from the side of C_n upon which $w=x-ij$, is seen to be the same as (23), except that $-j$ occurs throughout instead of j. We have then but to recall that as j increases indefinitely $\sinh(-\pi j)$ becomes infinite like $e^{\pi j}$ while $|\Gamma(x+k-ij)|$ vanishes to as high an order as that of $e^{-\pi j/2}$, to see that this contribution also vanishes in the limit as $j\to\infty$.

Next, let us consider the contribution from the side of C_n upon which $w=1/2+2n+iy$. Here we have $dw=idy$ and $\sin\pi w=\cos i\pi y=\cosh\pi y$, so that, having taken $j=\infty$ as indicated above, the contribution in question becomes

* See for example formula (10), page 24 of Nielsen's *Handbuch der Theorie der Gammafunction*. The formula there occurring, however, contains a typographical error inasmuch as the letter u which appears under the radical should be v.

$$J = \frac{(-z)^{1/2+2n}}{2} \int_{-\infty}^{\infty} \frac{g(2n+\frac{1}{2}+iy)(-z)^{iy}}{\Gamma(2n+k+\frac{1}{2}+iy)\cosh \pi y}\, dy$$
$$= \frac{z^{1/2+2n}}{2} \int_{-\infty}^{\infty} \frac{g(2n+\frac{1}{2}+iy)e^{iy\log\rho}}{\Gamma(2n+k+\frac{1}{2}+iy)\cosh \pi y}\, dy.$$

That the improper integral here appearing has a meaning follows when we recall that as y increases indefinitely in the positive sense, $\Gamma(2n+k+1/2+iy)$ vanishes to as high an order as that of $e^{-\pi y/2}$, and that as y increases indefinitely in the negative sense, the same function vanishes to as high an order as that of $e^{\pi y/2}$; also that, by virtue of (b), $g(2n+1/2+iy)$ remains less than a constant whatever the value of y. Moreover, we may now show that $\lim_{n\to\infty} J = 0$. In fact, this appears by writing J in the form

$$J = \frac{g(2n)z^{2n}}{\Gamma(2n+k)} \cdot \frac{z^{1/2}}{2} \int_{-\infty}^{\infty} \frac{g(2n+\frac{1}{2}+iy)}{g(2n)} \cdot \frac{\Gamma(k+2n)}{\Gamma(2n+k+\frac{1}{2}+iy)} \cdot \frac{e^{iy\log\rho}}{\cosh \pi y}\, dy,$$

and recalling that, inasmuch as the series (2) has an infinite radius of convergence, we have

$$\lim_{n\to\infty} \frac{g(2n)z^{2n}}{\Gamma(2n+k)} = 0,$$

whatever the value of z; also recalling our hypothesis (b) and the well known properties of the Γ function.

If we now take account of the contribution to the integral in (21) arising from the remaining side $w = a - 1/2 + iy$ of C_n, noting that we here have $dw = i\,dy$, $\sin \pi w = (-1)^{a-1}\cosh \pi y$ while the integration takes place from $y = +\infty$ to $y = -\infty$, the fundamental relation (21) takes the following form (n having been allowed to increase indefinitely as indicated above):

$$(24) \qquad \frac{(-1)^a(-z)^{a-1/2}}{2} \int_{-\infty}^{\infty} \frac{g(a-\frac{1}{2}+iy)(-z)^{iy}}{\Gamma(a+k-\frac{1}{2}+iy)\cosh \pi y}\, dy = \sum_{n=a}^{0} \frac{g(n)z^n}{\Gamma(n+k)} + f(z).$$

Thus far we have confined z to values which are real and negative. Suppose now that z as appearing in (24) is allowed to take on complex values. The right side of (24) is evidently analytic for all such values of z. We shall now show that the left side is likewise an analytic function of z so long as $R(z)<0$; that is, so long as $-3\pi/2<\phi<-\pi/2$. To establish this it suffices to show that the improper integral in (24) converges uniformly for values of z confined to such a region.* Now, the denominator of the integrand becomes infinite to as high an order as that of $e^{\pi y/2}$ when $y\to+\infty$, and it becomes infinite to as high an order as that of $e^{-\pi y/2}$ when $y\to-\infty$. As to the factor $(-z)^{iy}$, we have by (22)

$$(-z)^{iy} = e^{iy[\log \rho+i(\phi+\pi)]} = e^{iy \log \rho} e^{-(\phi+\pi)y},$$

and under the proposed restriction as regards ϕ, that is, $-3\pi/2<\phi<-\pi/2$, we have $-3\pi/2+\epsilon\leqq\phi\leqq-\pi/2-\epsilon$, $\epsilon>0$, and hence $-\pi/2+\epsilon\leqq\phi+\pi\leqq\pi/2-\epsilon$. We now have but to recall again hypothesis (b) of part II of the theorem to see that, under the indicated restrictions as regards ϕ, the integrand of (24) vanishes to as high an order as that of $e^{-\pi y/2}e^{-(-\pi/2+\epsilon)y}=e^{-\epsilon y}$ when $y\to+\infty$, and vanishes to as high an order as that of $e^{\pi y/2}e^{-(\pi/2-\epsilon)y}=e^{\epsilon y}$ when $y\to-\infty$. The uniform convergence property in question now follows immediately.

Inasmuch as (24) holds true, as we have now shown, for all negative values of z and at the same time each side is an analytic function of z so long as $R(z)<0$, it follows that it is likewise true for all such values of z. The proof of the theorem is thus complete.

3. *Generalizations and Remarks.* In order to secure simplicity of statement in the theorem the conditions (a) and (b) of part II were made unnecessarily restrictive. In fact, if one examines the proof of this part it appears readily that the following generalizations and remarks can be made.

---

* See for example Osgood, Encyklopädie der mathematischen Wissenschaften, II, 2, §6.

(α) In condition (b) the statement "remains less than a constant" may be replaced by the following stronger statement: "is such that, to an arbitrarily small positive quantity η there corresponds a positive constant K (independent of x and y) such that for all large values of $|x|$ and for all sufficiently large positive values of y one may write

$$\left|\frac{g(x \pm iy)}{g(x)}\right| < Ke^{\eta y}."$$

(β) In case condition (a) is not satisfied but instead the function $g(w)$, while still single valued throughout the finite w-plane, has m ($m \geqq 1$) singularities situated at the points $w = w_1, w_2, w_3, \cdots, w_m$, then (5) continues to hold true provided one subtracts from the right member the sum of the residues of the function

$$\frac{\pi g(w)(-z)^w}{\Gamma(w+k)\sin \pi w} \tag{25}$$

at these points.

(γ) In case condition (a) is not satisfied but instead the function $g(w)$, while still finite throughout the finite w-plane, is multiple valued owing to the presence of p ($p \geqq 1$) branch points at $w = W_1, W_2, W_3, \cdots, W_m, \cdots, W_p$, then (5) continues to hold true provided that one adds to the right member the sum of the p loop integrals of (25) of which the mth is described as follows: Draw the straight line extending from the point W_m to infinity in the direction of the positive (upper) half of the pure imaginary axis and let this line be regarded as a cut in the w-plane. Now let the loop consist of the two lines drawn on either side of and close to this cut, the ends of these lines near to the point W_m being joined by a circular arc of small radius drawn about this point. Moreover, let the integration over the loop as thus formed be understood to be in the positive (counter-clockwise) direction.

In case the exceptions described in (β) and (γ) both occur in $g(w)$ at the same time, the alterations to be made in (5) will then consist of both those described above.

Finally, it is to be noted that the theorem gives no information concerning the behavior of $f(z)$ in case $R(z)=0$. Naturally, this case calls for further investigation. We shall, however, omit it here, reserving it for possible discussion in a later paper.

4. *Application to the Function $G_\beta(z, \theta)$.* As stated in §1, we shall show in conclusion how the foregoing theorem, subject to the remarks in §3, may be applied in particular to determine the asymptotic behavior of the function $G_\beta(z, \theta)$ studied by Barnes and defined by (1).

Wishing first to apply part I, we write (1) in the form

$$G_\beta(z,\theta) = \sum_{n=0}^{\infty} \frac{g(n)z^n}{\Gamma(n+\beta+1)},$$

where

$$g(n) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^\beta}. \tag{26}$$

Now, it is a familiar fact immediately deducible from Stirling's formula for $\log\Gamma(x)$, that for values of x whose real part is positive, we have

$$\Gamma(x) \sim (2\pi)^{1/2}x^{x-1/2}e^{-x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right),$$

where $a_1, a_2, \cdots$ are constants. Hence

$$\Gamma(n+\beta+1) \sim (2\pi)^{1/2}e^{-(n+\beta+1)}(n+\beta+1)^{n+\beta+1/2} \cdot \left(1+\frac{a_1}{n+\beta+1}+\frac{a_2}{(n+\beta+1)^2}+\cdots\right),$$

$$\Gamma(n+1) \sim (2\pi)^{1/2}e^{-(n+1)}(n+1)^{n+1/2}\left(1+\frac{a_1}{n+1} + \frac{a_2}{(n+1)^2}+\cdots\right),$$

and therefore

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \sim e^{-\beta}\frac{(n+\beta+1)^{n+\beta+1/2}}{(n+1)^{n+1/2}}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right),$$

where b_1, b_2, b_3, etc. are constants whose values in terms of a_1, a_2, a_3, etc. may be readily determined. It follows that we may write

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^{\beta}} \sim e^{-\beta}\left(1+\frac{\beta-\theta+1}{n+\theta}\right)^{\beta} \cdot\left(1+\frac{\beta}{n+1}\right)^{n+1/2}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right).$$

The factor

$$\left(1+\frac{\beta-\theta+1}{n+\theta}\right)^{\beta}$$

here appearing is developable for large values of n in a convergent series proceeding according to ascending powers of $1/n$ while the factor

$$\left(1+\frac{\beta}{n+1}\right)^{n+1/2}$$

is developable for similar values of n in a convergent series of the form

$$e^{\beta}\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\frac{c_3}{n^3}+\cdots\right).$$

It follows that we may write

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^{\beta}} \sim 1+\frac{d_1}{n}+\frac{d_2}{n^2}+\frac{d_3}{n^3}+\cdots,$$

where d_1, d_2, d_3, etc. are assignable constants. Hence, we may also write

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^{\beta}} \sim 1+\frac{e_1}{n+\beta+1} + \frac{e_2}{(n+\beta+1)(n+\beta+2)}+\cdots, \tag{27}$$

where e_1, e_2, e_3, etc. are assignable constants. But (27) is the form (3) wherein $k=\beta+1$.

Applying (4), *we therefore may write when* $R(z)>0$

$$(28) \qquad G_\beta(z,\theta) \sim e^z z^{-\beta}\left(1+\frac{e_1}{z}+\frac{e_2}{z^2}+\frac{e_3}{z^3}+\cdots\right).$$

As to the values of the coefficients e_1, e_2, e_3, etc., they may evidently be computed to any desired point as follows, it being understood that $g(n)$ is defined by (26) and that the symbol Δ has its customary significance as used in the theory of finite differences; namely, $\Delta u(n)=u(n+1)-u(n)$:

$$(29) \quad \begin{cases} e_1 = -\lim\limits_{n\to\infty}(n+\beta+1)(n+\beta+2)\Delta g(n), \\ e_2 = +\lim\limits_{n\to\infty}(n+\beta+3)(n+\beta+4)\Delta[(n+\beta+1) \\ \qquad \cdot (n+\beta+2)\Delta g(n)], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Moreover, these limits are not difficult to evaluate when one makes repeated use of the relation $\Gamma(n+1)=n\Gamma(n)$.

Wishing now to apply part II of the theorem to the present function $G_\beta(z,\theta)$, we note that we here have the series (2) in which $k=1$ and $g(n)=(n+\theta)^{-\beta}$. As the function $g(w)$ to be used in applying the theorem, we therefore naturally take

$$(30) \qquad g(w) = (w+\theta)^{-\beta}.$$

This function at once satisfies condition (b) of the theorem so long as β is a number having a positive, or zero, real part. In case β has a negative real part the function

$$\frac{g(x\pm iy)}{g(x)} = \left(\frac{x+\theta\pm iy}{x+\theta}\right)^{-\beta} = \left(1\pm\frac{iy}{x+\theta}\right)^{-\beta}$$

evidently satisfies the condition mentioned in (α) of §3. In either case, therefore, the theorem may be applied so far as condition (b) is concerned. As regards condition (a), this will not be satisfied inasmuch as the function (30) will

have a singular point in general at $w=-\theta$. However, to meet this situation we need only to apply either remark (β) or (γ) of §3 according as the singular point in question is a pole (in which case β is a positive integer) or a branch point.

Applying (5), subject to the remarks (α), (β), (γ) of §3, and noting also that inasmuch as $k=1$ in the present instance, all the coefficients of (5) are equal to zero, *we conclude that when* $R(z)<0$ *we may write*

$$G_\beta(z,\theta) \sim L(z) + 0 + \frac{0}{z} + \frac{0}{z^2} + \cdots, \text{ or simply } G_\beta(z,\theta) \sim L(z),$$

where $L(z)$ *represents the negative of the residue of the function*

$$\frac{\pi(-z)^w}{\Gamma(w+1)(w+\theta)^\beta \sin \pi w}$$

at the point $w=-\theta$ *in case* β *is an integer, or represents the loop integral of this function about the same point in the manner described in* (γ) *of* §3 *in case* β *is not an integer.*

Results (28) and (29) are essentially those arrived at by Barnes for the special function $G_\beta(z, \theta)$.* However, he obtains the coefficients e_1, e_2, e_3, etc. under a different form of definition from (29), and through an elaborate investigation he evaluates the loop integral referred to above as $L(z)$.

THE UNIVERSITY OF MICHIGAN

* See result (A), page 265, vol. 206A (1906) of the Philosophical Transactions of the Royal Society of London.

A CORRECTION

BY W. L. AYRES

Dr. G. T. Whyburn has kindly called to my attention the fact that Theorems IV and V of my paper, *Concerning the boundaries of domains of a continuous curve*,* are incorrect. The following example constructed by Whyburn shows that theorem IV is false and a somewhat similar example may be used for Theorem V. Let K denote the rectangle with vertices $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ together with its interior. Let C_{-1} and C_0 be the circles with centers $(0, -1)$ and $(0, 1)$ and radii 1 and $\frac{1}{3}$ respectively. For each positive integer n let C_n be the circle with center $(1/2^n, (-1)^n)$ and radius $5/2^{n+3}$. Let C' and C'' be the set of all points (x, y) of $\sum_{i=-1}^{\infty} C_i$ for which $y>1$ and $y<-1$ respectively. Let $M=K+C'+C''$. Let D_{-1} be the set of all points (x, y) of K for which $x<0$ and $y<0$. Let D_0 be the set of all points (x, y) of K for which $x>\frac{1}{2}$. For each positive integer n let D_n be the set of all points (x, y) of K such that $1/2^{2n}>x>1/2^{2n+1}$. Let

$$D = C'' + \sum_{i=-1}^{\infty} D_i$$

and let P be the point $(-1, 1)$. The hypothesis of theorem IV is satisfied but one of the maximal connected subsets of the M-boundary of D with respect to P consists of the interval from $(0, 1)$ to $(0, -1)$ together with the interval from $(-1, 0)$ to $(0, 0)$.

I have found that Theorems IV and V are true if *either* of the following conditions be added to the hypotheses:

(4) *If B_1 is any maximal connected subset of B containing more than one point, then B_1 contains two non-cut-points which are accessible from both D and R.*†

* This Bulletin, vol. 33 (1927), pp. 565–571.

† The maximal connected subset of $M-\bar{D}$ containing P is denoted by R.

(4′) *If B_1 is any maximal connected subset of B containing more than one point, then the set of non-cut-points of B_1 is accessible from either D or R.*

If condition (4) be added, then these theorems may be proved by practically the same argument that I used. If condition (4′) be added, these theorems may be established by the following argument. The argument follows my "proof" of Theorem IV through line 7 page 569. As x and y are accessible from either D or R (we will suppose D), there is an arc xuy which lies in D except for x and y. Let T be a maximal connected subset of B_1-xzy. Suppose T has a limit point t on $<xzy>$. It is not difficult to see that T contains a non-cut-point s of B_1. There exists an arc st which lies in T except for the point t.* By (4′) there exists an arc rs which lies in D except for s and has only the point r in common with xuy. Now the set $xzy+xuy+rs+st$ divides the plane into three domains, one of which contains R. But if H is the domain containing R there is a point Q of B_1 which belongs neither to H nor to its boundary. Then Q cannot be a limit point of R. Hence every limit point of T belongs to the set $x+y$ and, since neither x nor y is a cut-point of B_1, both x and y are limit points of T. Then there is an arc xvy which lies in T except for x and y. Since every point of $xvy+xzy$ is a limit point of R, R lies in the complementary domain of $xuy+xvy+xzy$ which has $xvy+xzy$ as its boundary. Then the exterior of $xzy+xvy$ contains either D or R and the interior contains the other. Then $B=xzy+xvy$.

THE UNIVERSITY OF TEXAS

* See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), Theorem 1, p. 342.

SEVERI ON ALGEBRAIC GEOMETRY

Trattato di Geometria Algebrica. Volume I, Part I: *Geometria delle Serie Lineari.* By F. Severi. Bologna, 1926. viii+358 pp.

It is impossible to understand the principles on which a serious mathematical work is constructed unless one is in possession of a key to the author's thought. This should be found in the introduction, that is what an introduction is for, and in the present case Professor Severi sets forth his guiding principle in no uncertain terms.

"Con questo volume inizio la pubblicazione di un ampio Trattato, nel quale,—se il tempo e le forze mi basteranno—vorrei raccogliere, coordinare, completare dove occorra, tutto quanto vi è d'importante nel campo della geometria algebrica."*

To collect, coordinate and complete, when necessary, everything of importance in the field of algebraic geometry. This is the vast design which the author has set before himself. Consequently in the choice of subject matter and method, preference is given to that which has most connection with the general plan rather than to that which is immediately applicable to special problems.

The sub-heading of this first volume tells us that it deals with linear series, which means linear series of point groups on an algebraic curve. Why give so much attention to this particular topic? One is tempted to answer at once, "Because Professor Severi is an Italian, and this is an Italian specialty, almost a monopoly." That is no answer at all. A second answer would be because of the close connection with the study of algebraic functions and their integrals. But in the whole volume of 350 pages there is not a sign of integration, and scarcely any reference to a Riemann surface. All that will be very fully treated in subsequent volumes, but it is still far distant. What then? We must consider the group. The author in the present volume is concerned with algebraic curves, and with those properties of such curves which are unaltered by a general birational transformation. Such a transformation will carry a linear series into a linear series, it will preserve the genus of the curve and the structure of the Weierstrass points. The theory of linear series is the trunk whence all the branches of this birational geometry emanate. For instance, any non-hyperelliptic curve of genus p can be birationally transformed into a non-singular curve of order $2p-2$ in a space of $p-1$ dimensions, and this again can be carried into a plane curve if desired. Consequently in studying the birationally invariant properties of a curve, we may disregard the dimensionality of the space wherein it lies, and imagine that it is plane, or else skew and non-singular as we prefer. Herein we have a fundamental point

* P. vii.

of divergence from Professor Severi's first book which concerned itself primarily with plane curves.*

The first chapter is devoted to a study of linear systems of algebraic plane curves. This is a very large subject, but the author is concerned only with linear systems of curves as a means of reaching linear systems of point groups, and so leaves most of the topics on one side. What is essential is to establish the following theorem: An irreducible r-parameter system of algebraic curves is linear when, and only when, a single curve passes through r generic points, and the general curve is counted only once. An equal importance attaches to the beautiful theorem of Bertini: The general curve of a linear system will have a singular point only when this is a fixed singularity for all curves of the system.

Linear systems of curves lead to linear systems of point groups. Here, however, we may rise to as many dimensions as we please. Take an algebraic curve in a space of N dimensions. Cut it by a system of hypersurfaces linearly dependent on $r+s+1$ of their number, which are linearly independent, but of which s contain the curve, the result will be a linear r-parameter series of point groups, a g_n^r, on our curve. We may, once for all, leave the s hypersurfaces out of our consideration. The most general algebraic curve in a space of $r+s$ dimensions may be written

$$\rho x_i = f_i(x, y), \qquad F(x, y) = 0.$$

If the system of hypersurfaces be written

$$\sum_{i=0}^{i=r} \psi_i(x_0 \cdots x_N) = 0$$

we have effectively

$$F(x,y) = 0, \quad \sum_{i=0}^{i=r} \phi_i(x,y) = 0,$$

that is to say, the intersections of a plane curve by the curves of a linear system. Conversely, if we write

$$X_i = \phi_i(x, y), \qquad F(x, y) = 0,$$

and if the series be not composite, i.e., if the generic curve ϕ through a point of F does not automatically pass through another point of F variable with the first, why then our original curve is birationally transformed to a plane curve, and this birationally to another space curve where the series is cut by the hyperplanes of the space. The problem of linear series may thus be treated either as a problem *in plano*, or else as a problem of series cut by hyperplanes. The whole art consists in the skillful alternation of these two points of view.

At this point the author encounters his first serious difficulty, to explain the meaning of a branch of a curve. He turns it with great skill. First he shows how to transform a curve birationally into a space curve which has no singular points. This may be projected down to a non-singular curve in 3-space, and this again to a plane curve whose only singularities are nodes. The complete vicinity of any point of this curve is given by one,

* *Vorlesungen über Algebraische Geometrie*, Leipzig, 1921, a translation and amplification of his previous lithographed lectures.

or at most two, power series developments for x and y in terms of an auxiliary parameter t. Hence the complete vicinity of any point of the first curve is given by a finite number of such developments. If there be ν of these for a given point we shall say that there are ν branches there. How do we know that this number ν does not depend upon the method of transformation? To begin with we may show that if two curves are birationally related the exceptional points in the transformation are singular points. Hence if two non-singular space curves are birationally related there are no exceptional points. Hence the number of branches at a singular point is the number of corresponding developments on any non-singular space curve.

I must confess to being struck with admiration at the author's ingenuity in treating the problem of branches in this fashion. A better treatment for his purpose could not be devised. But the treatment, like all things human, has the defects of its qualities. No geometer has reached the heart of the singular-point difficulty, till he has made a thorough study of the power series developments.

Suppose that we have a linear series on a curve, a g_n^r, an r-parameter series of groups of n points. Certainly $n \geqq r$, for there can not be more parameters than points. Is it possible that our series is contained in a $g_n^{r'}$, where $r' > r$ and this in a $g_n^{r''}$, where $r'' > r$, etc.? Yes, this is certainly possible, but we must come to an end some time. Very well, when we have a series which is not contained in another of the same order but larger dimension, it is said to be "complete." How do we know that a given series is not contained in two different complete series? The usual proof is by the classical residuation theorem, the "Restsatz" of Brill and Nöther. But Severi, at this point, has none of the necessary apparatus at his disposal. He therefore turns to arithmetical and analytical considerations. Two groups of the same number of points are said to be "equivalent" if they are groups of the same series. Sums and differences of groups are defined and the laws of addition and subtraction established. If two series have a common group they are both contained in a larger series. Hence there can be but one complete series containing a given series, for if there were two, both would be contained in a third larger. I will hazard the guess that Severi must have regretted the very slight introduction of the zeros and poles of an algebraic function that he needed to prove this theorem, as a breach in the uniformity of his method; it was a very small breach at most.

In Chapter IV we reach the Jacobian group of a series, the group of multiple points, and a proof that this varies continuously with the series. This leads to the strange theorem, which requires simple transcendental considerations, that if we have two one-dimensional series, the difference between the Jacobian group and twice a general group of one, is equivalent to the corresponding difference in the other. There exists thus a single birationally invariant series which includes every such difference. Let the order of this invariant series be $2p-2$. Then p is defined as the "genus" of the curve and easily proved to be a positive integer when not zero. An

immediate application is made in finding the group of points of multiplicity $r+1$ of a g_n^r, Brill's formula.

We have at last all the essentials for the theory of linear groups, order, dimension, completeness, deficiency, canonical series, all present, all deduced by very rapid and simple operations. Anyone who is familiar with the subject will be struck by the beauty of it all. How will it impress the beginner? Perhaps he will feel the same way about it. But I cannot escape the feeling that he may find difficulty in understanding the motivation of the different steps. Why does the juggler take a hat from the table? Because he wishes to pull a rabbit out of it? But why a rabbit? Because well-born rabbits come out of hats? If the reader has this feeling at all, it will be intensified in the next chapter, the fifth, which is called the geometry of linear series, "secondo il metodo rapido." It certainly is rapid. Given an algebraic plane curve which is irreducible. Let $p+1$ be the smallest number of generic points such that the complete series including their group has not the dimension 0. Then p is defined as the genus of the curve and is shown to be zero when, and only when, the curve is rational. If the order of a complete generic series be $>p$, then $n-r=p$. If $n-r<p$, the series is said to be "special." These facts being birationally invariant, we may confine ourselves to the consideration of curves with no singularities but nodes. We then introduce the idea of adjoint curves, residual groups, and so the Riemann-Roch theorem and its consequences by the classical processes of Brill and Nöther. It is beautifully done and in a chapter of 25 pages the reader becomes acquainted with all the important theorems about linear series. But if one has learned elsewhere to associate the genus of a curve with a shortage of singular points, this new definition seems to be dragged in by the heels. Still, the result is beautiful.

In Chapter VI, the author turns to the theory of algebraic correspondence between curves and upon one curve. This is the longest and, it seems to me, the most important chapter in the book. There is a vast amount of literature upon the subject, the names of Chasles, Cayley, Brill and Zeuthen come at once to mind. No one can get to the very bottom of this theory without the use of abelian integrals, but the introduction through linear series which Severi exhibited in previous writings and which he gives here in full, is very far ahead of any other with which I am familiar.

The chapter begins with a study of birational transformations of a curve into itself, which leads to the study of projective transformations of a curve of higher space into itself. These are finite in number when the genus is above unity. We thus get the idea of moduli, or birationally invariant numbers connected with a curve. More generally, suppose that we have an algebraic transformation of a plane curve into itself. This may always be written

$$f(x, y)=f(x', y')=0,$$
$$\phi_1(x, y, x', y')=\phi_2(x, y, x', y')=0.$$

Castelnuovo determined an upper limit for the number of coincidences of such a correspondence and showed that it is reached when, and only when, there is need of but one auxiliary equation, $\phi_1=0$.

Here the groups of points (x', y') corresponding to a variable (x, y) are, with (x, y) itself counted γ times, the groups of a linear series. We call $\gamma<0$ the *value* of the correspondence. When positive it is an integer. On the other hand, if P correspond to $P_1 \cdots P_\nu$ and Q to $Q_1 \cdots Q_\nu$ and if $(-\gamma)$ be such a positive integer that the groups $P_1+P_2+\cdots+P_\nu+(-\gamma)Q$ and $Q_1+Q_2+\cdots+Q_\nu+(-\gamma)P$ are equivalent, we say that the correspondence has the negative integral value γ.

It may be shown by transcendental methods which Severi does not reproduce that on a curve of general moduli there are no correspondences but those of integral or zero value.

The value of the sum of two correspondences is the sum of their values, the value of the product, i.e., of the succession of two correspondences, is the negative of the product of their values. A pencil of curves will cut a correspondence of value unity. Hence we may construct a correspondence of any positive or negative integral value.

If we have a ν to ν' correspondence of value 0, the number of coincidences, which arise from the coalescence of two points on the same branch, is $\nu+\nu'$. Severi* proves this with great care, or rather proves the broader theorem that the sum of the groups corresponding to a given point in such a correspondence and its inverse is equivalent to the group of self-corresponding points. When it comes to correspondences of positive or negative value, we have merely to piece them out to correspondences of value zero by adding known correspondences of opposite value. We thus reach the Chasles-Cayley-Brill correspondence value

$$\nu+\nu'+2p\gamma.$$

In reaching this number, great care must be taken to count multiple coincidences correctly. The rule was found by Zeuthen and is carefully explained by Severi†; he also shows that this number is birationally invariant.

The value of a correspondence leads naturally to the important idea of the linear dependence of correspondences. The chapter closes with the discussion of another characteristic number of a correspondence called its "grade" which depends upon the coincidences between the correspondence and another differing infinitesimally from its inverse. I am willing to take the author's word that this is important, but must confess to finding it a bit obscure.

The great theory of the algebraic correspondence of curves developed in Chapter VI was based on the theory of linear series. The reader would naturally expect further developments in the subsequent chapters. Not at all; the author goes back to his original theme and exhibits two other methods of building up the theory "ab initio." In Chapter VII this is done by the methods of projective geometry in higher space, first brought to perfection by Segre and Castelnuovo, or Castelnuovo and Segre; it would be hard to settle the question of priority. The first thing is to find the complete series containing a given series. The essential point is to show that if

* Pp. 221 ff.

† Pp. 224 ff.

two linear series have a common group, they are contained in a series of higher dimension. The plan consists in studying groups on curves in spaces of r and r' dimensions and the ruled variety in $r+r'+1$ dimensions generated by the lines connecting the two, thus getting eventually a more ample system of spaces of $r+r'$ dimensions than that originally given. The next step consists in proving once more the Riemann-Roch theorem, and this is done by means of a formula already established for the number of groups of $r+1$ points common to a g_n^r and a g_m^1. The chapter is short and sketchy.

In the final chapter, Severi develops the theory of linear series by what he calls the "metodo algebrico." He takes the various steps in the following order:

1. A short account of Cremona transformations.
2. Study of the particular case of quadratic transformations.
3. A discussion of Nöther's idea of clusters of infinitely near singularities and a demonstration that any singular point can be treated as such a cluster.
4. A proof that the first polar of a general point is an adjoint curve whether the singularities are distinct or clustering. This shows that the number of clustering singularities must be finite. The proof that the form of the cluster is independent of the type of transformation used cannot be given satisfactorily without a study of the exponents in the power series developments.
5. A rather laborious proof that the known singularities of a curve impose independent conditions upon another curve of sufficiently high order.
6. Nöther's $AF+B\phi$ theorem.
7. Definition of adjoint curves, and residue theorem. Demonstration that the adjunction conditions are independent.
8. Riemann-Roch theorem as before.

It is part of a reviewer's duty to put the reader on guard against the writer's predilections and prejudices; a candid reviewer should also give warning against his own personal preferences. And the present reviewer must confess to a feeling of regret that the book was not written backwards, with this algebraic method explained at the outset, and the geometric methods added subsequently. This classical procedure which has been followed by Brill and Nöther and Bertini seems to me to penetrate more completely into the heart of the fundamental questions involved, especially when joined to a study of the series developments. Such a study will surely be made in subsequent volumes. The author sees the whole edifice which he plans to build; the reviewer can only see in part, and so he must make the mistake of judging the part separated from the whole. When shall we see the whole? That secret is still in the lap of the gods. The author says in the preface: "Con questo volume inizio la publicazione di un Trattato—se il tempo e le forze mi basteranno....." Every reader will hope ardently that they do.

J. L. COOLIDGE

SHORTER NOTICES

Algebra. By Oskar Perron. Berlin and Leipzig, Walter de Gruyter, 1927. Two volumes. Volume I. *Die Grundlagen.* 4 figures; viii+307 pp. Volume II. *Theorie der algebraischen Gleichungen.* 5 figures; viii+243 pp. Göschens Lehrbucherei, I. Gruppe: Reine Mathematik, Band 8, 9.

In view of the large number of texts on higher algebra already on the market, the average mathematician is not apt to hail a new book on this subject with joy or even interest. As one's eye runs hastily down the table of contents and sees a sequence of titles that calls to mind a composite picture of Weber, Serret, Burnside and Panton, et al., one can certainly be excused if one does not at first wax enthusiastic. In fact, the chapter-headings in Volume I are, in order: fundamental notions; polynomial theorem and Taylor's theorem; determinants (including the elements of matrices and of bilinear, quadratic and Hermitian forms); symmetric functions; divisibility of polynomials; existence of roots; and in Volume II the chapter-headings are: numerical solution of equations; reciprocal equations and equations of degree $\geqq 4$; substitution groups; Galois' theory of equations; equations of degree 5. Moreover, the captions of the subdivisions of the eleven chapters are, generally, the classic ones. Thus, first impressions lead one to expect merely one more book to add to the library catalogue and to which a beginning graduate student will look for help when Weber, Serret et al. happen to be in use. But, fortunately, first impressions are unjust in this case. For there are two places in the book where the treatment is sufficiently different from the classic treatment to interest a reader familiar with the usual treatise on the subject.

Probably the outstanding characteristic of the book is the prominence which is given to the notion of field (domain of rationality). This is certainly good pedagogy, for this is one of the most important concepts in mathematics. To be sure, the most central notion of all algebra is that of a linear algebra (hypercomplex numbers) including as special cases fields, groups, matrices and integral equations. But it would be too much to expect the author of a comparatively elementary text to display his various topics as special cases or applications of linear algebras; and we are glad to see that he has clustered the more strictly algebraic parts of his book around the notion of field. In particular, he has presented the subject of the divisibility of polynomials in one or more variables in such a way that there are not those annoying exceptions in the statement of the theorems about the highest common factor of two polynomials in two or more variables, as in Bôcher's text. Also, the emphasis on fields in the first volume admirably prepares the way for the discussion of Galois' theory of equations in the second volume. Although the notion of field is always used in this subject, of necessity, yet the author's presentation is such as to give to the student a number of important theorems about algebraic fields which he does not usually see formulated outside a course in algebraic numbers

and should give him a grasp of the elements of this subject such as he does not usually have after taking the usual course in Galois' theory of equations. This aspect of the book is highly to be commended.

On the other hand, when the author has departed to this extent from the precedent set by Weber, Serret et al., it is all the more disappointing to find that he includes the classic material on the numerical solution of equations—Sturm's theorem, the theorem of Fourier and Budan, Descartes' rule of signs, upper limits of the roots, Newton's method, "regula falsi," and the methods of approximation due to Bernoulli and Gräffe, although he is wise enough to omit Horner's method. Mathematics of this kind certainly does not belong in the same book with a neat and clear exposition of the divisibility of polynomials in several variables and Galois' theory of equations except in so far as it may be used to illustrate a bit of theory. In fact, such mathematics is not algebra at all, but what one might call "algebraic statistics"—that is, it applies to concrete problems in algebra the methods of statistics rather than the methods of pure algebra.

O. C. HAZLETT

Sur la Théorie des Equations aux Dérivées Partielles du Premier Ordre d'une Seule Fonction Inconnue. By N. N. Saltykow. Paris, Gauthier-Villars, 1925. 172 pp.

This book contains the lectures delivered by the author in the four Belgian Universities during 1923–1924. Much of the material represents results of the author's own researches which have not previously been published in detail or which are now presented in simplified form.

The treatment centers about the well known theorem concerning the equivalence of the two problems: (1) to find a complete integral of a given partial differential equation of the first order in one unknown function; (2) to find the general integral of the equations of its characteristics. The author develops a theory which exhibits the relations between these problems and which gives methods of finding a solution of either one when any incomplete set of integrals of the equations of the characteristics is known. He extends his results, moreover, to simultaneous systems by proofs like those for the single equation. In fact, the similarity is such that one wonders why he does not treat the more general case at the outset.

Since the characteristics can be defined by a linear homogeneous system of partial differential equations, there is an introductory chapter on the theory of such systems. In particular, a new form of the equations of the characteristics, fundamental for the subsequent developments, results from this preliminary discussion.

The contributions are both interesting and important. The details of the exposition could perhaps be improved upon. A clear-cut statement of hypothesis and conclusion would help. For example, at the beginning of Chapter II the author announces he is about to prove Jacobi's theorem, but leaves the reader to find out which one. The misprints are numerous but of little consequence.

J. M. THOMAS,
National Research Fellow

The Mechanical Investigations of Leonardo da Vinci. By I. B. Hart. Chicago, Open Court Publishing Company, 1925. 2+240 pp.

This book is a very interesting account of a chapter in the history of mechanics, centering around the picturesque figure of Leonardo da Vinci. Leonardo's fame as the painter of The Last Supper and Mona Lisa has tended to cause an undervaluation of the excellent pioneer work which he did in the field of statics and dynamics. Dr. Hart has here assembled in good form the most important of these results, and has added a translation of the manuscript of Leonardo on the Flight of Birds, the only complete translation into English, so far as I know, of a manuscript of Leonardo.

There are eight chapters: I, pp. 1–12, a general discussion of Leonardo's manuscripts; II, pp. 13–35, mechanical science in the 15th century (largely devoted to the instruments used); III, pp. 36–53, contemporary scientific influences; IV, pp. 54–74, Leonardo's sources; V, pp. 75–98, Leonardo's work in dynamics; VI, pp. 99–142, Leonardo's work in statics; VII, pp. 143–193, Leonardo as a pioneer of aviation; VIII, pp. 194–235, translation of Leonardo's manuscript on the flight of birds. There is a good index.

To give some idea of the scope of Leonardo's work, a few of the results that are brought together in Dr. Hart's book may be mentioned. In dynamics, Leonardo stated the principle of inertia, and that of the equality of action and reaction, at least in special cases. He anticipated Galileo in stating the law for the velocity acquired by a falling body, though he did not succeed in obtaining the correct result for the distance fallen, thinking it, like the velocity, to be proportional to the time. Similar results are stated for bodies rolling down an inclined plane; but here, as also in his attempts at the composition and resolution of forces, he was much handicapped by lack of knowledge of trigonometry, which science was not yet generally known to Europeans, as Regiomontanus's great treatise, though existing in manuscript, was not published till much later (1533).

In statics, Leonardo determines certain centers of gravity, including (perhaps for the first time) that of the pyramid; studies the principles of the lever and pulley, giving nearly all the combinations that modern textbooks discuss; shows how power may be transmitted by various types of gearing; and considers the problem of loaded beams.

The last two chapters are the most important part of the book, since Leonardo's study of the flight of birds forms one of the most significant and valuable of the attempts of men to learn the secret of flight. We find here very careful observations of birds and the various motions of their wings, also the action of the tail as a rudder; and clever reasoning on the center of gravity and center of pressure, and the mechanical principles involved, including the case of soaring flight. The observations and discussions are illustrated by fine drawings.

Thus Leonardo da Vinci is seen to have been a pioneer in the development of aviation; and Dr. Hart's book, which makes these results available to the English reader, is accordingly a valuable addition to the literature of the history of science.

R. B. McClenon

Le Calcul des Probabilités—Son Évolution Mathématique et Philosophique. By L.-Gustave du Pasquier. Paris, Hermann, 1926. 21+304 pp., two tables.

Du Pasquier has written a delightful book in which the classic theory of probability centering around the so-called Theorem of Bernoulli is given in some detail, with an adequate setting in history, logic, and philosophy. Well chosen examples illustrate the use of the probability integral—for which two tables are given in the appendix—but the author is not interested primarily in technique. Nor has he found space for a mathematical treatment of generalized frequency functions, although he mentions the work of Pearson and others, and admits (p. 258) the inadequacy of the Gaussian function for biology. One of his chief purposes is to trace historically the ideas underlying the concept of probability—ideas very hazy at first, clarified to some extent by J. von Kries who insisted upon a "cogent reason" for cases declared to be "equally likely," made more lucid by means of the theory of ensembles used to determine "zones de comportement" (p. 217), and given a finishing touch by R. von Mises* who by the use of an infinite sequence as a "Kollektiv" with its accompanying "Verteilung" eliminates "equally likely cases" as a primary idea.

In Chapter V, six interpretations of probability are unfolded, designated: psychological, practical, logical, empirical, inductive, and interpretation by the zones of comportment. The view is held (pp. 188, 197, Chap. VI) that objective probability or "chance" exists, independent of human knowledge. Furthermore, even complete knowledge does not destroy probability—contrary to the view of Lourié (p. 203) that probability is the science for systematizing ignorance. The examples chosen to support Du Pasquier's contention are taken from the seemingly fortuitous behavior of numbers such as those forming the rth decimal place of the logarithms of consecutive integers. But later (p. 284) the author finds that such examples do not involve the "irregularity" demanded by the Second Postulate of von Mises—so they cannot be admitted to full standing as a "collectif," but must rank as a "syllepte" (p. 286).

The interesting applications in Chapter VII to the kinetic theory of gases, reversible and irreversible phenomena, entropy, etc., have a certain philosophic aroma. We are told (p. 229) that the phenomenon of fluctuation in the density of a gas adds a temperament to the inflexible determinism which rules the material universe.

Chapter VIII is devoted to the exposition of the theory of R. von Mises, setting forth the two postulates which determine a Kollektiv, and explaining the simple operations for deriving cumulative frequency functions from given cumulative frequency functions, "Verteilungen." This is well written, —in the statement of Postulate II, however (p. 266), there appears "ne soient pas nulles" instead of "nicht beide null." The concluding Chapter IX

---

* *Grundlagen der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 5 (1919), pp. 52–99.

on mathematical probability and experience discusses further the postulates of von Mises and also logical systems.

The book, throughout, is written in an entertaining style, free from many details that would be uninteresting to the average reader. Although the reviewer was unable to verify the formula in the middle of page 123 and the one at the foot of page 128, the book seemed to be exceptionally free from misprints and infelicities. It will be welcomed by those who are interested in the foundations of probability.

E. L. DODD

Curve Sghembe Speciali Algebriche e Trascendenti. Volume II: *Curve Sferiche, Curve Definite da una Relazione fra Flessione e Torsione, Curve Particolare situate sopra Superficie Assegnate.* By Gino Loria. Bologna, N. Zanichelli, 1925. 255 pp.

The second volume of the treatise on special space curves* treats both algebraic and transcendental curves. Those having tangents belonging to a linear complex base are discussed at length, followed by an outline of those belonging to quadratic and higher complex. Differential properties and methods of proof are particularly featured. Over a fifth of the volume is devoted to spherical curves; it is fairly exhaustive and is well written. Another fifth is given to curves defined by intrinsic equations. The last and longest chapter discusses curves on given surfaces, including helices, lines of curvature, geodesics, and asymptotic lines. The application of the latter to ruled surfaces contained in linear congruences does not take account of a number of important articles.

Extensive references are given, and a list of all the authors quoted in both volumes is given at the end. This feature is a particularly valuable one for bibliographic purposes. The proof reading has been very well done, except that German titles in the footnotes must occasionally suffer.

VIRGIL SNYDER

La Série de Taylor et son Prolongement Analytique. By J. Hadamard and S. Mandelbrojt. Scientia, No. 41. Deuxième édition, revue et mise au courant des progrès récents. Paris, Gauthier-Villars, 1926. 104 pp.

The systematic study of the singularities of analytic functions was begun by Hadamard. In 1901, a very valuable account of his own investigations together with those of other early workers, as Fabry, Leau, LeRoy, Borel and others, was presented by Hadamard in his now classic little book *La Série de Taylor et son Prolongement Analytique* published in the Collection Scientia (No. 12).

This work has now been revised and brought up to date by Hadamard, with the assistance of the brilliant young mathematician Mandelbrojt, who has published in the last few years a number of valuable papers bearing on the subject. In this edition, the authors present in addition to the

* The first volume was reviewed in the Bulletin, vol. 31 (1925), p. 557.

results of the original edition all recent researches treating of the singularities and the analytic continuation of analytic functions defined by a Taylor's series, and, in the words of the authors, "presenting a real interest."

On account of the amount of new material incorporated in this new edition, the work is being issued in two volumes, the first of these two volumes having been published in 1926, while the second volume is in preparation. This first volume contains a revised treatment of the subject matter of the first five chapters making up the first half of the original work, together with the newer researches relating to these topics. The second volume will contain a similar treatment of the other half of the original book. The bibliography has been greatly extended and brought up to date, the original 36 references of the first edition bearing on the first five chapters having now been increased to 91. All mathematicians interested in the study of analytic functions and their singularities will surely welcome this revised edition of a classic in its field.

L. L. SMAIL

La Méthode de Darboux et les Equations $s=f(x, y, z, p, q)$. Mémorial des Sciences Mathématiques, No. 12. By R. Gosse. Paris, Gauthier-Villars, 1926. 52 pp.

The first three chapters of this work are devoted almost exclusively to summarizing results and methods developed more at length in Chapters 4, 6, 7, and 8 of Goursat's *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Second Ordre*. Chapter I concerns the theory of characteristics for the general partial differential equation of the second order in one unknown and two independent variables. It contains a neat proof of the uniqueness of the solution of Cauchy's problem for such an equation. Chapter II deals with invariants of the given equation and with equations of order n in involution with it. In addition to a short exposition of Darboux's method, Chapter III contains some theorems, due to Gau, on the formation of invariants and their reduction to a canonical form.

The last two chapters discuss the problem of determining what equations of the particular type mentioned in the title of the book are integrable by Darboux's method. An assemblage of results contained chiefly in papers by Gau, Goursat, and the author shows that all such equations which do not have an intermediate integral can be reduced by simple transformations to standard types. Some results are also given for two special forms of the equation having an intermediate integral.

The treatment is of necessity very sketchy. For this reason the general mathematical reader will doubtless prefer a standard treatise when seeking information about Darboux's method. The present pamphlet should, therefore, make its appeal to specialists in the field treated in its last two chapters. To them it should be valuable because it contains a synopsis of the present state of the subject, though its usefulness in this respect is apt to be impaired by the absence of a systematic bibliography.

J. M. THOMAS,
National Research Fellow

NOTES

The July, 1927, number of the Transactions of this Society (volume 29, No. 3) contains the following papers: *The contact of a cubic surface with an analytic surface*, by E. P. Lane; *Dynamical space-times which contain a conformal euclidean 3-space*, by H. P. Robertson; *Reduction of the ordinary linear differential equation of the nth order whose coefficients are certain polynomials in a parameter to a system of n first-order equations which are linear in the parameter*, by C. E. Wilder; *On the "third axiom of metric space,"* by V. W. Niemytzki; *Implicit functions and differential equations in general analysis*, by L. M. Graves; *Concerning acyclic continuous curves*, by H. M. Gehman; *On a general formula in the theory of Tchebycheff polynomials and its applications*, by J. Shohat; *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i z}$, by J. F. Ritt; *Arithmetic of logic*, by E. T. Bell; *Functionals of r-dimensional manifolds admitting continuous groups of point transformations*, by A. D. Michal. The October, 1927, number contains: *The differential equation of the elliptic cylinder*, by J. H. McDonald; *On a class of integral equations with discontinuous kernels*, by R. E. Langer and E. P. Brown; *Some third-order irregular boundary value problems*, by L. E. Ward; *Concerning point sets which can be made connected by the addition of a simple continuous arc*, by G. T. Whyburn; *The Notion of Green's function in the theory of integro-differential equations*, by J. D. Tamarkin; *A general theory of nets on a surface*, by V. G. Grove; *Simply transitive primitive groups*, by W. A. Manning; *The expansion problems associated with regular differential systems of the second order*, by M. H. Stone; Errata, volumes 21, 28, 29.

The July, 1927, number of the American Journal of Mathematics (volume 49, No. 3) contains the following papers: *Linear ordinary self-adjoint differential equations of the second order*, by Anna Pell Wheeler; *The singularities of a function defined by a Dirichlet series*, by D. V. Widder; *On entire function interpolation*, by I. M. Sheffer; *Optics in space of constant non-vanishing curvature*, by J. Pierpont; *On an imprimitive group of order* 5184, by J. R. Musselman; *Rational involutorial transformations in* S_4 *which leave invariant* ∞^4 *quadric varieties*, by H. C. Shaub; *A three-dimensional quartic variety in four-space*, by B. C. Wong; *Compound singularities of the rational plane quartic curve*, by J. H. Neelley; *Pencils of conics in the Galois fields of order* 2^n, by A. D. Campbell; *On generalized lacunae*, by J. J. Gergen; *Properties of certain aggregate functions*, by L. S. Hill; *The theory of group-reduced distributions*, by J. H. Redfield; *Closure of the tangential process on the rational plane cubic*, by F. E. Allen.

At the meeting of the Society to be held in Chicago on April 6–7, Professors E. P. Lane and E. B. Stouffer will be the principal speakers at a symposium entitled Recent developments in projective differential geometry.

The following schedule of lectures has already been arranged for Professor Constantin Carathéodory, of the University of Munich, who will be the first Visiting Lecturer of this Society. (See page 22 of this number of this Bulletin.) The University of Pennsylvania, January 10–13; Ohio State University, January 16–17; The University of Iowa, January 19–20; The University of Chicago, January 23–24; The University of Michigan, January 25–27; Adelbert College, January 30–31; Cornell University, February 1–2. After these lectures, Professor Carathéodory will be in residence at Harvard University until about June 1. Arrangements for additional lectures will be announced later.

The Lobachefsky prize of the Physico-Mathematical Society of Kazan has been awarded to Professor Hermann Weyl, of the Zurich Technical School, for his work as a whole and in particular for his contributions to the problems of space from the point of view of the theory of groups and for his researches on the representation of continuous groups. Earlier awards of this prize were to Lie in 1897, Killing in 1900, and Hilbert in 1903.

Cambridge University has awarded its John Winbolt prize in engineering jointly to C. Hinton, of Trinity College, and W. R. D. Manning, of Sidney Sussex College, for a dissertation on *Transverse oscillation of bridges*.

The Royal Academy of Madrid has announced the following subject for a prize memoir for 1927: *An exposition of the fundamental principles of analysis situs and its most important applications*; competition restricted to the Spanish, Portuguese, and Spanish Americans.

The Reale Accademia dei Lincei has elected Professor Giovanni Georgi, of the University of Cagliari, a correspondent in the section of mechanics, and Professor N. E. Nörlund, of the University of Copenhagen, a foreign member in the section of mathematics.

Professor Rudolf Mehmke has received the honorary degree of doctor of engineering from the Stuttgart Technical School, on the occasion of his seventieth birthday.

The University of Bristol, on the occasion of the opening of its new physics laboratory, has conferred honorary doctorates on Professor Max Born, of Göttingen, Sir William Bragg, of the Royal Institution, London, Professors A. S. Eddington, of Cambridge, Alfred Fowler, of the Imperial College of Science and Technology, and Paul Langevin, of Paris, and Sir Ernest Rutherford, of Cambridge.

The University of Toronto has conferred an honorary doctorate of science on Professor Louise D. Cummings, of Vassar College.

Dr. Paul Alexandroff, of Moscow, and Dr. Heinze Hopf, of Berlin, have been granted International Education Board fellowships for 1927–28, for the purpose of becoming acquainted with the work of American mathematicians in the field of analysis situs. They will study at Princeton and Harvard.

The following have been appointed National Research Fellows in mathematics for 1927–28 (this list includes reappointments): W. L. Ayres, R. W. Barnard, Alonzo Church, C. M. Cramlet, Jesse Douglas, D. A. Flanders, Orrin Frink, G. M. Merriman, T. W. Moore, Hillel Poritsky, H. P. Robertson, C. F. Roos, I. M. Sheffer, M. M. Slotnick, W. M. Whyburn.

Dr. F. Hund, of Göttingen, has been called to an associate professorship of theoretical physics at the University of Rostock.

Professor Robert König, of the University of Münster, has been appointed professor of mathematics at the University of Jena.

Dr. Maximilian Krafft has been appointed associate professor of mathematics at the University of Marburg.

Professor E. Salkowski, of Hannover, has been called to the professorship of descriptive geometry and practical mathematics at the Charlottenburg Technical School.

Dr. Erich Schönhardt has been appointed associate professor of mathematics at the University of Tübingen.

Professor Alois Timpe, of the Berlin Agricultural School, has been appointed professor of mathematics at the Charlottenburg Technical School.

The following have been admitted as private docents: Dr. F. K. Schmidt, in mathematics, at the University of Erlangen; Frau Hilda Pollaczek-Geiringer, in applied mathematics, at the University of Berlin.

Dr. Peter Debye, professor of physics at the Zurich Technical School, has accepted a call to the University of Leipzig, as successor to Professor Otto Wiener.

Dr. W. Saxer has been appointed professor of descriptive geometry at the Zurich Technical School.

Professor Giulio Pittarelli, of the chair of descriptive geometry at the University of Rome, has retired.

Miss Gwenthalyn Jones, of Chicago, has presented Princeton University with $200,000 for the endowment of a professorship of mathematical physics; the chair will be named after her uncle, Thomas D. Jones, of the Princeton class of 1876.

Dr. Ethel L. Anderton has been promoted to an assistant professorship of mathematics at Smith College.

Dr. Tobias Dantzig, professor of mathematics at the University of Maryland, will conduct a course in advanced mathematics for physicists and chemists at the Bureau of Standards during the year 1927–28.

Professor H. C. Gossard, of the University of Wyoming, has been appointed professor of mathematics and dean of men at Nebraska Wesleyan University.

Dr. Harold Hotelling, research associate in the Food Research Institute until the fall of 1927, has been appointed associate professor of mathematics at Stanford University.

Professor F. R. Moulton, director of the department of astronomy at the University of Chicago, has resigned, to become associated in an executive capacity with the Utilities Power and Light Corporation of Chicago.

Dr. B. C. Patterson has been appointed associate professor of mathematics at Hamilton College.

Associate Professor J. B. Reynolds, of Lehigh University, has been promoted to a full professorship of mathematics and theoretical mechanics.

Miss Georgia E. Robinson has been appointed professor of physics at Ozark Wesleyan College.

Dr. G. T. Whyburn has been appointed assistant professor of mathematics at the University of Texas.

The following appointments to instructorships in mathematics are announced.

Amherst College, Mr. B. LeF. Brown;
University of Chicago, Mr. T. F. Cope.

Professor Svante Arrhenius, director of the Nobel Institute, Stockholm, died October 2, 1927, at the age of sixty-eight.

Miss A. L. Van Benschoten, formerly professor of mathematics at Wells College, died September 18, 1927. Professor Van Benschoten had been a member of the American Mathematical Society since 1903.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

AUBRY (A.). See BALL (W. W. R.).

BALDUS (R.). Nichteuklidische Geometrie. Hyperbolische Geometrie der Ebene. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 152 pp.

BALL (W. W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. 2e édition française. 3e partie, avec additions de MM. A. Margossian, Reinhart, J. Fitzpatrick, et A. Aubry. Paris, Hermann, 1927. 363 pp.

BARZIN (M.) et ERRERA (A.). Sur la logique de M. Brouwer. Bruxelles, 1927. 16 pp.

BECKER (O.). Mathematische Existenz. Untersuchungen zur Logik und Ontologie mathematischer Phänomene. Halle, Niemeyer, 1927. 7+369 pp.

BUCHANAN (H. E.) and SPERRY (P.). Plane trigonometry and tables. Richmond, Johnson, 1926. 11+116+112 pp.

CARMICHAEL (R. D.) and WEAVER (J. H.). The calculus. Boston, Ginn, 1927. 14+345 pp.

CLAREN (J.). Neue Methode zur Lösung von kubischen Gleichungen zum Selbststudium. 2te erweiterte Auflage. Leipzig, Verlag Bange, 1927. 20 pp.

DELAMBRE (—.). See PTOLÉMÉE.

DESCARTES (R.). La géométrie. Nouvelle édition. Avec portrait de Descartes d'après Frans Hals. Paris, Hermann, 1927. 91 pp.

DREETZ (W.). See SCHÜLKE (A.).

ERRERA (A.). Exposé historique du problème des quatre couleurs. Bologna, 1927. 24 pp.

——See BARZIN (M.).

EUWE (M.). Differentiaalinvarianten van twee covariante-vectorvelden met vier veranderlijken. (Dissertation, Amsterdam.) Groningen, Noordhoff, 1926. 85 pp.

FINE (H. B.). Calculus. New York, Macmillan, 1927. 8+421 pp.

FITZPATRICK (J.). See BALL (W. W. R.).

GAUSS (C. F.). Gesammelte Werke. Band XI, Abteilung 1: Varia, Atlas des Erdmagnetismus, Biographisches Register. Berlin, Springer, 1927. 518 pp.

GAUSS (C. F.) und GERLING (C. L.). Briefwechsel. Herausgegeben im Auftrage der Gesellschaft zur Beförderung der gesamten Naturwissenschaften zu Marburg von C. Schaefer. Berlin, Otto Elsner Verlagsgesellschaft, 1927. 20+820 pp.

GERLING (C. L.). See GAUSS (C. F.).

HALMA (—.). See PTOLÉMÉE.

HASSE (H.). Höhere Algebra. II: Gleichungen höheren Grades. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 160 pp.

HAZARD (C. T.). See MASON (T. E.).

HOBSON (E. W.). The theory of functions of a real variable and the theory of Fourier's series. 3d edition, revised throughout and enlarged. Volume 1. Cambridge, University Press, 1927. 16+736 pp.

HORN (J.). Gewöhnliche Differentialgleichungen. 2te, völlig umgearbeitete Auflage. (Göschens Lehrbücherei.) Berlin, de Gruyter, 1927. 8+197 pp.

KEASEY (M. A.), KLINE (G. A.), and MCILHATTEN (D. A.). Plane trigonometry, with tables. Philadelphia, Blakiston, 1927. 7+130 pp.

KLINE (G. A.). See KEASEY (M. A.).

KRAITCHIK (M.). Le problème du cavalier. Bruxelles, l'Echiquier, 1927. 5+91 pp.

LITTLE (A. S.). A table of interpolation multipliers. London, Routledge, 1927.

MCILHATTEN (D. A.). See KEASEY (M. A.).

MARGOSSIAN (A.). See BALL (W. W. R.).

MASON (T. E.) and HAZARD (C. T.). Analytic geometry. Boston, Ginn, 1927. 11+224 pp.

PASCAL (E.). Introduzione al calcolo infinitesimale. Napoli, 1926. 4+116 pp.

PASCH (M.). Mathematik am Ursprung. Gesammelte Abhandlungen über Grundfragen der Mathematik. Leipzig, Meiner, 1927. 7+149 pp.

PICARD (E.). Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique. Paris, Gauthier-Villars, 1927. 214 pp.

POTRON (—.). Exercices de calcul différentiel et intégral. Volume 2: Solution des exercices. Paris, Hermann, 1927. 4+258 pp.

RUSSELL (B.). The analysis of matter. London, Kegan Paul, and New York, Harcourt, Brace and Company, 1927. 8+408 pp.

PTOLÉMÉE. Composition mathématique. Traduite pour la première fois du grec en français par M. Halma et suivie des notes de M. Delambre. Tome 1 et tome 2. Paris, 1813, 1816. Réimpression fac-similé, Paris, Hermann, 1927. 76+476+48+8+448+40 pp.

REINHART (—.). See BALL (W. W. R.).

SCHAEFER (C.). See GAUSS (C. F.).

SCHÜLKE (A.) und DREETZ (W.). Aufgabensammlung aus der reinen und angewandten Mathematik. Leitfaden der Mathematik. Leipzig, Teubner, 1926–27.

SCHUH (F.). Beknopte hoogere algebra. Tweede stuk. Groningen, Noordhoff, 1926. 287 pp.

SEVERI (F.). Trattato di geometria algebrica. Volume I, parte I. Bologna, Zanichelli, 1926. 358 pp.

SPERRY (P.). See BUCHANAN (H. E.).

VICI (G.). Sulla sezione aurea di un segmento. Milano, Redaelli, 1926. 23 pp.

WEAVER (J. H.). See CARMICHAEL (R. D.).

WIELEITNER (H.). Mathematische Quellenbücher. Band 1: Rechnen und Algebra. Berlin, Salle, 1927. 7+75 pp.

PART II. APPLIED MATHEMATICS

BOUASSE (H.). Verges et plaques, cloches et carillons. Paris, Delagrave, 1927. 480 pp.

BOUTARIC (A.). La physique moderne et l'électron. Paris, Alcan, 1927. 266 pp.

BRAY (F.). Light. London, Arnold, 1927. 12+284+6 pp.

BURGESS (R. W.). Introduction to the mathematics of statistics. Boston, Houghton Mifflin, 1927. 10+304 pp.

CAVEN (R. M.). Atoms and molecules. Being part I and chapter XII of The foundations of chemical theory. London, Blackie, 1927. 8+141 pp.

ERIKSON (H. A.). Elements of mechanics. New York, McGraw-Hill, 1927. 16+150 pp.

FELDMANN (C.). Die Berechnung elektrischer Leitungsnetze in Theorie und Praxis. Berlin, Springer, 1927. 10+554 pp.

FENDER (W.). See MOULTON (F. R.).

FRANK (P.). See RIEMANN (B.).

GEHRCKE (E.), herausgegeben von. Handbuch der physikalischen Optik. 2ter Band, 2te Hälfte, 1ter Teil. Leipzig, Barth, 1927. 388+6 pp.

GEIGER (H.) und SCHEEL (K.), herausgegeben von. Handbuch der Physik. Band 5: Grundlagen der Mechanik. Mechanik der Punkte und starren Körper. Redigiert von R. Grammel. Band 7: Mechanik der flüssigen und gasförmigen Körper. Redigiert von R. Grammel. Berlin, Springer, 1927. 14+623+11+413 pp.

GOCHT (H.). Die Röntgen-Literatur. Stuttgart, Enke, 1927. 12+515 pp.

GRAMMEL (R.). See GEIGER (H.).

GRÖBER (H.). Einführung in die Lehre von der Wärmeübertragung. Berlin, Springer, 1926. 10+200 pp.

GUYOT (J.). See LEMOINE (J.).

HARRISON (H. H.). Elements of telephone transmission. New York, Longmans, 1927.

HARROW (B.). The romance of the atom. New York, Boni and Liveright, 1927. 162 pp.

HEDGES (E. S.) and MYERS (J. L.). The problem of physico-chemical periodicity. London, Arnold, 1926. 95 pp.

HOUSTOUN (R. A.). A treatise on light. 5th edition. London, Longmans, 1927. 489 pp.

KELEN (N.). Die Staumauern. Theorie und wirtschaftliche Bemessung. Berlin, Springer, 1926. 8+294 pp.

LEECHMAN (G. F.). The theory and practice of steering. Glasgow, J. Brown, 1927. 102 pp.

LEMOINE (J.) et GUYOT (J.). Cours de physique. Tome III: Magnétisme, électricité. Paris, Vuibert, 1927. 362 pp.

LEWIS (G. N.). Die Valenz und der Bau der Atome und Moleküle. Übersetzt von G. Wagner und H. Wolff. Braunschweig, Vieweg, 1927.

MACMILLAN (W. D.). Theoretical mechanics. Statics and the dynamics of a particle. New York, McGraw-Hill, 1927. 18+430 pp.

MAYER (M.). Nomographie des Bauingenieurs. (Sammlung Göschen.) Berlin, de Gruyter, 1927.

MERKEL (F.). Die Grundlagen der Wärmeübertragung. Dresden, Steinkopf, 1927. 11+234 pp.

MILLIES (A.). Räumliche Vieleckrahmen mit eingespannten Füssen unter besonderer Berücksichtigung der Windbelastung. Berlin, Springer, 1927. 6+96 pp.

MOULTON (F. R.). Einführung in die Himmelsmechanik. 2te, durchgesehene Auflage. Autorisierte deutsche Ausgabe von W. Fender. Leipzig, Teubner, 1927. 13+412 pp.

MÜLLER-BRESLAU (E.). La scienza delle costruzioni. Traduzione italiana di C. Rossi e L. Santarella. Volume III: Sistemi staticamente indeterminati (parte seconda). Milano, Hoepli, 1927. 14+798 pp.

MYERS (J. L.). See HEDGES (E. S.).

NIKURADSE (J.). Untersuchung über die Geschwindigkeitsverteilung in turbulenten Strömungen. Berlin, Verein Deutscher Ingenieure, 1926.

PALMER (L. S.). Wireless principles and practice. New York, Longmans, 1927.

POHL (R. W.). Einführung in die Elektrizitätslehre. Berlin, Springer, 1927. 7+256 pp.

PRASIL (F.). Technische Hydrodynamik. 2te, erweiterte Auflage. Berlin, Springer, 1927.

REICH (F.). Umlenkung eines freien Flüssigkeitsstrahles an einer senkrecht zur Strömungsrichtung stehenden Platte. Berlin, Verein Deutscher Ingenieure, 1926. 152 pp.

REJTÖ (A.). Einige Prinzipien der theoretischen mechanischen Technologie der Metalle. Berlin, Verein Deutscher Ingenieure, 1927. 21+501 pp.

RIEMANN (B.) und WEBER (H.). Die Differential- und Integralgleichungen der Mechanik und Physik. 2ter, physikalischer Teil, herausgegeben von P. Frank. Braunschweig, Vieweg, 1927. 28+863 pp.

ROSSI (C.). See MÜLLER-BRESLAU (E.).

SANTARELLA (L.). See MÜLLER-BRESLAU (E.).

SCHEEL (K.). See GEIGER (H.).

TOLMAN (R. C.). Statistical mechanics with applications to physics and chemistry. (American Chemical Society Monograph Series.) 334 pp.

WAGNER (G.). See LEWIS (G. N.).

WEBER (H.). See RIEMANN (B.).

WICKE (F.). Einführung in die höhere Mathematik unter besonderer Berücksichtigung der Bedürfnisse des Ingenieurs. In zwei Bänden. Berlin, Springer, 1927. 10+921 pp.

WOLFF (H.). See LEWIS (G. N.).

Members of the American Mathematical Society will receive a discount of 25% on all of the publications listed below. The net prices to members are stated.

Mathematische Annalen—Mathematische Zeitschrift

One or two volumes of the ANNALEN, and two or three volumes of the ZEITSCHRIFT, appear each year. Each volume contains about 800 pages. The net price to members is $12.75 per volume, plus postage.

Die Grundlehren der mathematischen Wissenschaften

This series is edited by R. Courant, assisted by W. Blaschke and M. Born. Recent volumes are listed in this BULLETIN under New Publications. The net price to members of recent volumes of about 300 pages is $3.10 plus postage.

Orders for books of the Courant series, for the ANNALEN, and for the ZEITSCHRIFT, should be sent to the publisher, Julius Springer, Berlin, or his official agent, who is also an official agent for this Society:

Hirschwaldsche Buchhandlung, Unter den Linden 68, Berlin

Jahrbuch über die Fortschritte der Mathematik

For recent issues, the net prices to members, including mailing, are: vol. 46 (1916-18), $17.05; vol. 47, Nos. 1-5, $9.80; vol. 48, Nos. 1-3, $6.70. Orders should be sent to

L. Bieberbach, Marienbaderstr. 9, Berlin-Schmargendorf

or to

Walter de Gruyter & Co., Genthinerstr. 38, Berlin

This firm publishes also CRELLE'S JOURNAL, the MINERVA-ZEITSCHRIFT, and many other journals.

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität

One volume of about 400 pages appears each year. The net price to members is $3.90, plus a mailing charge, if orders are sent to:

B. G. Teubner, Publisher, Leipzig, Germany

NEW BOOKS

The Calculus

By ROBERT D. CARMICHAEL, *Professor of Mathematics,* University of Illinois, and JAMES H. WEAVER, *Professor of Mathematics,* Ohio State University.

A new first course in which presentation and organization have been guided by the psychological laws of learning. Price, $2.80.

Analytic Geometry

By THOMAS E. MASON, *Professor of Mathematics* and CLIFTON T. HAZARD, *Associate Professor of Mathematics,* Purdue University. With the editorial co-operation of ROBERT D. CARMICHAEL.

A new introductory course in which the student's acquisition of an effective method of attack on problems is of primary importance. Price, $2.40.

Boston **GINN AND COMPANY** **New York**
Chicago **Atlanta** **Dallas** **Columbus** **San Francisco**

We Are the Sole AMERICAN AGENTS

for

BELL'S ADVANCED MATHEMATICAL SERIES

First Course in Nomography by S. Brodetsky. Price **$3.00**

Projective Vector Algebra by L. Silberstein. Price **$1.75**

First Course in Statistics by D. Caradog Jones. Price **$3.75**

Elementary Treatise on Differential Equations and Their Application by H. T. H. Piaggio. Price **$3.50**

Elementary Vector Analysis with Application to Geometry and Physics by C. E. Weatherburn. Price **$3.50**

Advanced Vector Analysis by C. E. Weatherburn. Price **$3.50**

Mathematical Theory of Limits by J. G. Leathem. Price **$4.50**

Send for complete catalog

THE OPEN COURT PUBLISHING COMPANY
337 East Chicago Avenue, Chicago, Illinois

Mémorial des Sciences Mathématiques

Edited by HENRI VILLAT

Under the patronage of la Société mathématique de France, l'Académie des Sciences de Paris, and the scientific academies of Servia, Belgium, Bucarest, Poland, Ukraine, Spain, Prague, Rome (dei Lincei), and Sweden. See this BULLETIN, vol. 31, Nos. 5-6, p. 281.

The following seventeen monographs have appeared at 10 fr. each.

PAUL APPELL, *Sur une forme générale des équations de la dynamique.*

G. VALIRON, *Fonctions entières et fonctions méromorphes.*

PAUL APPELL, *Séries hypergéométriques de plusiers variables, polynomes d'Hermite et autres fonctions sphériques de l'hyperspace.*

M. D'OCAGNE, *Esquisse d'ensemble de la nomographie.*

P. LÉVY, *Analyse fonctionnelle.*

E. GOURSAT, *Le problème de Bäcklund.*

A. BUHL, *Séries analytiques. Sommabilité.*

TH. DE DONDER, *Introduction à la gravifique einsteinienne.*

E. CARTAN, *La géométrie des espaces de Riemann.*

P. HUMBERT, *Fonctions de Lamé et fonctions de Mathieu.*

G. BOULIGAND, *Fonctions harmoniques. Principes de Picard et de Dirichlet.*

R. GOSSE, *La méthode de Darboux pour les équations* $s=f(x, y, z, p, q)$.

A. VÉRONNET, *Figures d'équilibre et cosmogonie.*

TH. DE DONDER, *Théorie des champs gravifiques.*

S. ZAREMBA, *La logique des mathématiques.*

A. BUHL, *Formules Stokiennes.*

G. VALIRON, *Théorie générale des séries de Dirichlet.*

Many others are in press or in preparation.

For all information, address

Gauthier-Villars et Cie., 55, Quai des Grands-Augustins, Paris

SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, 0 means "zero"; to indicate "cap O," backspace and overprint period; thus: O.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) A B $\Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma} \cdots$ (all except $^{\iota}$ and ^{o}).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma} \ldots$ (all except $_{\iota}$ and $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \rho\ \vartheta\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q}$ $\mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}$ $\mathfrak{A\ B\ D}$

Script (special font $\mathcal{A\ B\ C} \cdots$ (all). No lower case manufactured.

* Hebrew— א בּ ב troublesome to handle.

★ Dashed Italics—$\overline{A}\ \bar{a}\ \overline{B}\ \bar{b}\ \overline{C}\ \bar{c}\ \overline{E}\ \bar{e}\ \overline{F}\ \bar{f}\ \overline{G}\ \bar{g}\ \overline{H}\ \overline{I}\ \bar{\imath}\ \overline{K}\ \bar{k}\ \overline{M}\ \bar{m}\ \overline{N}\ \bar{n}\ \overline{P}\ \bar{p}\ \bar{q}\ \overline{R}\ \bar{r}\ \bar{s}\ \bar{t}\ \bar{u}\ \bar{v}\ \overline{X}\ \bar{x}\ \overline{Y}\ \bar{y}\ \overline{Z}\ \bar{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\varepsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \overline{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

---

* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 25, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than February 4.

CHICAGO, April 6-7, 1928.

Abstracts must be in the hands of Professor M. H. Ingraham, Associate Secretary of the Society, University of Wisconsin, Madison, Wis., not later than March 16. At the request of the Program Committee, Professors E. P. Lane and E. B. Stouffer will read papers at a symposium entitled *Recent developments in projective differential geometry*.

NEW YORK CITY, April 6-7, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than March 16.

STANFORD UNIVERSITY, California, April 7, 1928. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than March 17.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, Menasha, Wis., or 501 West 116th Street, New York.

Advertising space is available in the BULLETIN at $16 per page, $9 per half page, $5 per quarter page, per issue. Address correspondence regarding advertising to H. L. RIETZ, University of Iowa, Iowa City, Iowa.

Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

WHOLE NUMBER 352

CONTENTS

	PAGE
The October Meeting in New York. By R. G. D. Richardson	1
The Fifty-Third Regular Meeting of the San Francisco Section. By B. A. Bernstein	15
The Twentieth Regular Meeting of the Southwestern Section. By E. B. Stouffer	18
The Visiting Lectureship of the American Mathematical Society	22
Mathematical Rigor, Past and Present. By James Pierpont	23
A Further Note on the Converse of Fermat's Theorem. By D. H. Lehmer	54
On the Approximate Representation of Analytic Functions. By Dunham Jackson	56
Generalizations of the Theorem of Fermat and Cauchy on Polygonal Numbers. By L. E. Dickson	63
A Rational Normal Form for Certain Quartics. By Raymond Garver	73
On the Law of the Excluded Middle. By Alonzo Church	75
Note on Interchange of Order of Limits. By T. H. Hildebrandt	79
On Bounded Regular Frontiers in the Plane. By W. A. Wilson	81
On the Behavior of Integral Functions in Distant Portions of the Plane. By W. B. Ford	91
A Correction. By W. L. Ayres	107
Severi on Algebraic Geometry. By J. L. Coolidge	109
Shorter Notices	115
Notes	121
New Publications	125

For official Communications and Notices, see the inside of the back cover.